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The injective hull of ultra-quasi-metric versus q-hyperconvex hull of quasi-metric space

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ABSTRACT

For any partially ordered set equipped with its natural T_0-quasi-metric (T_0-ultra-quasi-metric), we study the connection between the ultra-quasi-metrically injective hull and the q-hyperconvex hull. We also observe that for a partially ordered set, its Dedekind–MacNeille completion coincides exactly with the ultra-quasi-metric injective hull of its natural T_0-ultra-quasi metric.

1. Introduction

In [9] Künzi and Olela Otafudu presented the construction of the ultra-quasi-metrically injective hull in the category of T_0-ultra-quasi-metric spaces and contracting maps. Comparable studies have been done in the category of ultra-metric spaces and contracting maps by Bayod and Martínez-Maurica (see [2]). In the category of T_0-quasi-metric spaces and nonexpansive maps, the q-hyperconvex hull (Isbell-hull) has been studied by Kemajou et al. (see [7]). We should mention that similar investigations were presented in the category of metric spaces and nonexpansive maps independently by Isbell [6] and Dress [4]. While

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the q-hyperconvex hull of a T_0-quasi-metric space is q-hyperconvex, the ultra-quasi-metrically injective hull of a T_0-ultra-quasi-metric space is q-spherically complete. It is well-known that a q-spherically complete space does not need to be q-hyperconvex space. In this paper, for a partially ordered set endowed with its natural T_0-quasi-metric (which is of course T_0-ultra-quasi-metric), we study the connection between its q-hyperconvex hull and its ultra-quasi-metrically injective hull. In particular, we show that for a partially ordered set equipped with the natural T_0-quasi-metric (T_0-ultra-quasi-metric), its ultra-quasi-metrically injective hull is contained in its q-hyperconvex hull. Our investigations can be applied to study the similarity between the injective hull due to Isbell [6] and the injective hull due to Bayod and Martínez-Maurica [2]. We have to mention that recently in [5], Agyingi et al. proved that for a partially ordered set equipped with the natural T_0-quasi-metric space, its q-hyperconvex hull contains its Dedekind–MacNeille completion.

2. Preliminaries

This section recalls the most important definitions and preliminary results that we shall use in this paper.

Definition 1. Let X be a set and let $d : X \times X \to [0, \infty)$ be a function mapping into the set $[0, \infty)$ of the nonnegative reals. Then, d is called a *quasi-pseudometric* on X if

(a) $d(x, x) = 0$ whenever $x \in X$,

(b) $d(x, z) \leq d(x, y) + d(y, z)$ whenever $x, y, z \in X$.

We shall say that d is a T_0-*quasi-metric* provided that d also satisfies the following condition: for each $x, y \in X$,

$d(x, y) = 0 = d(y, x)$ implies that $x = y$.

Remark 1. Let d be a quasi-pseudometric on a set X, then $d^{-1} : X \times X \to [0, \infty)$ defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also a quasi-pseudometric, is called the *conjugate quasi-pseudometric of d*. As usual, a quasi-pseudometric d on X such that $d = d^{-1}$ is called a *pseudometric*. Note that for any (T_0-)quasi-pseudometric d, $d^s = \max\{d, d^{-1}\} = d \vee d^{-1}$ is a pseudometric (metric).

Remark 2. For a quasi-pseudometric space (X, d): we have

1. for each $x \in X$ and $\epsilon > 0$, $B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ denotes the *open ϵ-ball* at x,
2. the collection of all "open" balls yields a base for a topology $\tau(d)$. It is called the *topology induced by d* on X,
3. similarly we set for each $x \in X$ and $\epsilon \geq 0$, $C_d(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\}$. Note that $C_d(x, \epsilon)$ is $\tau(d^{-1})$-closed, but not $\tau(d)$-closed in general.

Remark 3. (compare [13, Definition 1]) A T_0-quasi-metric space is called *a T_0-ultra-quasi-metric space* if it satisfies the strong triangle inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}, \quad \text{whenever} \quad x, y, z \in X.$$

Example 1. ([10, Example 0.2.1]) Let $X = [0, \infty)$ be equipped with $n(x, y) = x$ if $x > y$ and $n(x, y) = 0$ if $x \leq y$. Then (X, n) is a T_0-ultra-quasi-metric space.

A map $f : (X, d) \to (Y, e)$ between two quasi-pseudometric spaces (X, d) and (Y, e) is called an *isometric map* provided that $e(f(x), f(y)) = d(x, y)$ whenever $x, y \in X$. Observe that if $f : X \to Y$ is an isometric map between two quasi-pseudometric spaces X and Y and if X is a T_0-quasi-metric space, then f is injective.

Two quasi-pseudometric spaces (X, d) and (Y, e) will be called *isometric* provided that there exists a bijective isometric map $f : (X, d) \to (Y, e)$.