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# Pair correlations of aperiodic inflation rules via renormalisation: Some interesting examples

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This article presents, in an illustrative fashion, a first step towards an extension of the spectral theory of constant length substitutions. Our starting point is the general observation that the symbolic picture (as defined by the substitution rule) and its geometric counterpart with natural prototile sizes (as defined by the induced inflation rule) may differ considerably. On the geometric side, an aperiodic inflation system possesses a set of exact renormalisation relations for its pair correlation coefficients. Here, we derive these relations for some paradigmatic examples and infer various spectral consequences. In particular, we consider the Fibonacci chain, revisit the Thue–Morse and the Rudin–Shapiro system, and finally analyse a twisted extension of the silver mean chain with mixed singular spectrum.



## 1. Introduction

The spectral theory of primitive substitution rules and the symbolic dynamical systems defined by them has a long history and is well studied, compare [34] and references therein. Nevertheless, as soon as one leaves the realm of constant length substitutions, many open questions challenge our present level of understanding. This is partly due to the fact that the substitution system picture and the inflation tiling picture, where one works with natural prototile lengths, may differ considerably; compare [15,35,5,8].

The majority of the dynamical systems literature on this subject deals with the symbolic case, where work by Dekking [17] and others, see [19–21,29] and references therein, has led to a fairly complete understanding of the constant length case, which was recently extended in a systematic fashion in [13], including higher dimensions. In all these cases, the connection between the diffraction spectrum and the dynamical spectrum is rather well understood. The two notions are equivalent in the pure point case [28,7,9,31], and recent progress for symbolic systems also establishes a complete equivalence via the inclusion of certain factors [10].

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Therefore, at least algorithmically, any constant length substitution can be analysed completely as far as its spectrum is concerned; see [3,6,13] for recent developments.

The situation is less favourable outside this class. Even if one stays within the realm of primitive inflation rules, the determination of the dynamical spectrum is generally difficult. Here, the geometric counterpart studied in tiling theory has certain undeniable advantages. First, since such tilings are still of finite local complexity, the connection between the dynamical spectrum and the diffraction spectrum can still be used, if certain factors are included in the discussion [10]. For this reason, we employ an approach via the diffraction spectrum of the tiling spaces. Second, the geometric setting leads to the existence of a set of *exact* renormalisation relations for the pair correlation coefficients of the system. We use the term 'exact' to distinguish our approach from the widely used renormalisation schemes that are approximate or asymptotic in nature. This approach has not attracted much attention so far. In fact, we are not aware of *any* reference to this approach beyond the constant length case, though the principal idea has certainly been around for a while, and has been used, often in an approximate way, for several physical quantities such as electronic or transport properties; see [41] for an example and further references.

In this paper, we want to show the power of exact renormalisation relations for spectral properties, in the form of a first step via some illustrative examples. A more general approach will be presented in a forthcoming publication. It will be instrumental for our analysis that we formulate various aspects on the symbolic level, while the core of our analysis rests on the natural geometric realisation. To make the distinction as transparent as possible, we will speak of *substitution rules* on the symbolic side, but of *inflation rule* on its geometric counterpart, thus following the notation and terminology of the recent monograph [5]. Also, various results are briefly recalled from there. Rather than repeating the proofs, we provide precise references instead.

Below, we work along a number of examples, all of them with Pisot–Vijayaraghavan (PV) numbers as inflation multipliers. Since we do *not* demand the characteristic polynomial to be irreducible over $\mathbb{Q}$, there is still enough freedom to encounter systems with mixed spectrum. In fact, all point sets that we encounter along the way will be linearly repetitive Meyer sets. An interesting linearly repetitive inflation point set with non-PV multiplier (hence not a Meyer set) will be discussed in detail in [2].

The paper is structured as follows. After recalling some facts about translation bounded measures on $\mathbb{R}$ and their Fourier transforms in Section 2, we begin with a detailed analysis of the Fibonacci substitution and inflation in Section 3. This is both a paradigm of the theory and an instructive example along which we can develop our ideas as well as further notions, wherefore this is also the longest section. Here, we introduce a set of exact renormalisation equations for the general pair correlation coefficients and compare the findings with what is known from the model set description. Then, we use the new approach to derive an alternative proof of the pure point nature of the diffraction spectrum, and hence also of the dynamical spectrum via the known equivalence result for this case [28,7,31].

After that, in Section 4, we briefly revisit the classic Thue–Morse and Rudin–Shapiro sequences from the renormalisation point of view. The point of this exercise is to show that our approach, in the constant length setting where the symbolic and the geometric pictures coincide, is essentially equivalent to the traditional approach as described in [34] and recently extended in [13]. Still, there are several aspects of a more algebraic nature that seem to deserve further attention.

Finally, by imposing an involutory *bar swap* symmetry, we construct an extension of the silver mean chain with mixed spectrum in Section 5. The main point here is that such extensions are not restricted to the constant length case (where they are known from examples such as those of the previous section or many others as in [3,4,6]). In fact, as our example indicates, there is an abundance of interesting and completely natural primitive inflation rules that produce repetitive Meyer sets with mixed spectrum. In our opinion, this has hitherto been more or less neglected. For the explicit analysis, the exact renormalisation scheme is used to determine the spectral type by a somewhat subtle application of the Riemann–Lebesgue lemma, which leads to a singular spectrum of mixed type.