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Geometrical and spectral properties of Pisot substitutions

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ABSTRACT

A necessary condition for an n-dimensional substitutive dynamical system to be a finite extension of an algebraic system is that the substitution be Pisot. We discuss the sufficiency of the Pisot property and review conditions under which such an extension is guaranteed to be almost everywhere one-to-one.

1. Introduction

In this article we consider tilings of n-dimensional Euclidean space generated by substitutions. Any such tiling gives rise to a pair of interacting dynamical systems on the 'hull', or 'tiling space', of the tiling consisting of the space of all tilings locally indistinguishable from the tiling. One of these systems is the $\mathbb{R}^n$-action by translation, the other is a hyperbolic action induced by the substitution. Similar systems arise in algebraic dynamics: consider the situation of a hyperbolic toral (or solenoidal) automorphism with $\mathbb{R}^n$-action being translation along unstable manifolds. The question we address here is: Under what conditions on the substitution are the tiling dynamical systems a finite (or almost everywhere one-to-one) extension of such algebraic systems?

A substitution is made of three ingredients: a finite collection of 'prototiles', which are labeled, compact and topologically regular, subsets of $\mathbb{R}^n$; a linear expansion Λ of $\mathbb{R}^n$; and a combinatorial rule that describes how each inflated (by Λ) prototile is tiled by translates of prototiles. It was first noted by Thurston [36] for $n = 2$ and then more generally by Kenyon [22], Kenyon and Solomyak [23], and finally Kwapisz [24] that

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the mere existence of such a combinatorial rule places severe restrictions on the inflation Λ – for example, all of its eigenvalues must be algebraic integers and if λ' is an algebraic conjugate of an eigenvalue λ of Λ with $|\lambda'| \geq |\lambda|$, then λ' must also be an eigenvalue of Λ. (The most complete description is that Λ must be 'integral Perron' – see [24].)

A primary motivation for the topics considered here comes from a fundamental question in mathematical crystallography: What self-affine point patterns $\Gamma \subset \mathbb{R}^n$ diffract? Substitutions provide a method for generating self-affine point patterns, the diffraction properties of which are closely related to the dynamical spectrum of the $\mathbb{R}^n$-action tiling dynamical system (for a survey, see [27]). For example, in dimension n, the point pattern generated by a substitution has pure point diffraction if and only if the ($\mathbb{R}^n$-action) tiling dynamical system has pure discrete dynamical spectrum. A necessary condition for the latter for dimension $n = 1$ is that the inflation $\Lambda = (\lambda)$ be a Pisot number (that is, λ is an algebraic integer greater than 1 all of whose algebraic conjugates lie strictly inside the unit circle). For arbitrary n, the diffraction pattern of a substitution-generated point set has a relatively dense set of Bragg peaks if and only if the tiling $\mathbb{R}^n$-action has a relatively dense set of 'continuous eigenvalues,' and the latter happens if and only if the $\mathbb{R}^n$-action on the tiling space is a finite extension of an action of $\mathbb{R}^n$, by translation, on a torus or solenoid. It is necessary and sufficient for the last condition that the tilings in the tiling space have the 'Meyer property' (in terms of an associated point set Γ, this means that $\Gamma - \Gamma$ is uniformly discrete) and we will see that a necessary (and, most likely, sufficient) condition for the Meyer property is that the inflation Λ be Pisot (an arithmetical condition generalizing Pisot number).

In the discussion above, sharp diffraction properties of a self-affine point pattern are correlated with the translation action on tiling space being a finite extension of translation on a torus (or solenoid). We will see that such an extension automatically also semi-conjugates the substitution action on tiling space with a hyperbolic automorphism of the torus (or solenoid). It is really the interaction of the two dynamical systems, translation and substitution, that is key to the results we present here.

This article is based on talks the author gave at the workshop titled 'The Pisot Conjecture,' which was held at the Lorentz Center in Leiden between September 1 and September 5, 2014. Quite a bit of the material, particularly of the last two sections, is repackaged from [9,8,16,13]. After a section covering notation and general properties of tilings, we consider the finite extension question and the relations between Pisot and Meyer. The penultimate section presents various conditions that imply, or are equivalent to, pure discrete spectrum for the translation action, and in the final section these conditions are applied to demonstrate pure discrete spectrum for a variety of one-dimensional families.

2. Notation and terminology

A (symbolic) substitution $\phi : \mathcal{A} = \{1, \dots, d\} \to \mathcal{A}^* :=$ {finite nonempty words in $\mathcal{A}$} has *abelianization*, or *incidence matrix* $M = (m_{ij})_{d \times d}$ with m_{ij} equal to the number of times the letter i appears in the word $\phi(j)$. We will only deal with *primitive* ϕ, that is, ϕ such that some power of M strictly positive. We will use λ to denote the Perron–Frobenius eigenvalue of M and call it *the inflation of* ϕ; a positive left eigenvector for M will be denoted by $\omega = (\omega_1, \dots, \omega_n)$. The *substitutive system* associated with ϕ is the subshift (X_ϕ, σ), where X_ϕ is the collection of all bi-infinite words $w = (w_i)_{i \in \mathbb{Z}}$ having the property that every finite factor (subword) of w occurs as a factor of $\phi^k(j)$ for some $k \in \mathbb{N}$, $j \in \mathcal{A}$, and σ denotes the shift: $\sigma((w_i)_i) = (w_{i+1})_i$. All substitutions ϕ we consider will be *non-periodic*, meaning that there are no periodic orbits in (X_ϕ, σ).

To associate tilings of $\mathbb{R}$ with ϕ we first define the collection of *prototiles*, $\{\rho_1, \dots, \rho_d\}$. Let $\rho_i := ([0, \omega_i], i)$: the interval $[0, \omega_i]$ is called the *support* of ρ_i (denoted $spt(\rho_i)$) and i is called the *type* of ρ_i. A translate $\tau = \rho_i - x := ([0, \omega_i] - x, i)$ is called a *tile* (its *support* is $[-x, \omega_i - x]$, its *type* is i, and tiles may be similarly translated) and a finite collection P of tiles is called a *patch* provided distinct tiles in P have disjoint interiors. The *support of* P is $spt(P) := \cup_{\tau \in P} spt(\tau)$. The *tile substitution* Φ associated with ϕ takes prototiles to patches by $\Phi(\rho_i) := \{\rho_{i_1}, \rho_{i_2} + \omega_{i_1}, \dots, \rho_{i_m} + \omega_{i_1} + \dots + \omega_{i_{m-1}}\}$, where $\phi(i) = i_1 \cdots i_m$. Note