Topology and its Applications 205 (2016) 47–57

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

The S-adic Pisot conjecture on two letters ☆

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ARTICLE INFO

Article history:
Received 26 January 2015
Received in revised form 23 July 2015
Accepted 23 July 2015
Available online 1 February 2016

MSC:
37B10
37A30
52C23
68R15

Keywords:
S-adic shifts
Substitutions
Tilings
Pisot numbers
Coincidences
Lyapunov exponents
Balancedness

ABSTRACT

We prove an extension of the well-known Pisot substitution conjecture to the S-adic symbolic setting on two letters. The proof relies on the use of Rauzy fractals and on the fact that strong coincidences hold in this framework.

1. Introduction

A symbolic substitution is a morphism of the free monoid. More generally, a substitution rule acts on a finite collection of tiles by first inflating them, and then subdividing them into translates of tiles of the initial collection. Substitutions thus generate symbolic dynamical systems as well as tiling spaces. The Pisot substitution conjecture states that any substitutive dynamical system has pure discrete spectrum, under

☆ The authors are participants in the ANR/FWF project "FAN – Fractals and Numeration" (ANR-12-IS01-0002, FWF grant I1136).

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http://dx.doi.org/10.1016/j.topol.2016.01.019

the algebraic assumption that its expansion factor is a Pisot number (together with some extra assumption of irreducibility). Pure discrete spectrum means that the substitutive dynamical system is measurably conjugate to a rotation on a compact abelian group. Pisot substitutions are thus expected to produce self-similar systems with long range order. This conjecture has been proved in the two-letter case; see [4] together with [16] or [17]. For more on the Pisot substitution conjecture, see e.g. [6,1].

We prove an extension of the two-letter Pisot substitution conjecture to the symbolic S-adic framework, that is, for infinite words generated by iterating different substitutions in a prescribed order. More precisely, an S-adic expansion of an infinite word ω is given by a sequence of substitutions $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ (called directive sequence) and a sequence of letters $(i_n)_{n\in\mathbb{N}}$, such that $\omega = \lim_{n\to\infty} \sigma_0\sigma_1 \cdots \sigma_n(i_{n+1})$. Such expansions are widely studied. They occur e.g. under the term "mixed substitution" or "multi-substitution" [15,20], or else as "fusion systems" [18,19]. They are closely connected to adic systems, such as considered e.g. in [12]. For more on S-adic systems, see [10,11,9,5].

We consider the (unimodular) Pisot S-adic framework introduced in [7], where unimodular means that the incidence matrices of the substitutions are unimodular. The S-adic Pisot condition is stated in terms of Lyapunov exponents: the second Lyapunov exponent associated with the shift space made of the directive sequences $\boldsymbol{\sigma}$ and with the cocycles provided by the incidence matrices of the substitutions is negative. For a given directive sequence $\boldsymbol{\sigma}$, let $\mathcal{L}_{\boldsymbol{\sigma}}^{(k)}$ stand for the language associated with the shifted directive sequence $(\sigma_{n+k})_{n\in\mathbb{N}}$. The Pisot condition implies in particular that, for some $C > 0$, the language $\mathcal{L}_{\boldsymbol{\sigma}}^{(k)}$ is C-balanced, for some infinite set of non-negative integers k. Recall that C-balancedness means that the number of occurrences of any letter in any factor of given length is the same up to some deviation bounded by C. We note that in [22] it is shown that balancedness characterizes topological conjugacy under changes in the lengths of the tiles of the associated $\mathbb{R}$-action on the 1-dimensional tilings.

One difficulty when working in the S-adic framework is that no natural candidate exists for a left eigenvector (that is, for a stable space). Recall that the normalized left eigenvector (whose existence comes from the Perron–Frobenius Theorem) in the substitutive case provides in particular the measure of the tiles of the associated tiling of the line. We introduce here a set of assumptions which, among other things, allows us to work with a generalized left eigenvector (that is, a stable space). We stress the fact that this vector is not canonically defined (contrary to the right eigenvector). These assumptions will be implied by the S-adic Pisot condition. We need first to guarantee the existence of a generalized right eigenvector $\mathbf{u}$ (which yields the unstable space). The corresponding conditions are natural and are stated in terms of primitivity of the directive sequence $\boldsymbol{\sigma}$ in the S-adic framework. This means that long enough products of incidence matrices associated with $\boldsymbol{\sigma}$ will be positive. We also require the directive sequences $\boldsymbol{\sigma}$ to be recurrent (every finite combination of substitutions in the sequence occurs infinitely often), which yields unique ergodicity and strong convergence toward the generalized right eigendirection. We then need rational independence of the coordinates of $\mathbf{u}$. This is implied by the assumption of algebraic irreducibility, which states that the characteristic polynomials of long enough products of incidence matrices of $\boldsymbol{\sigma}$ are irreducible. Lastly, we then will be able to define a vector playing the role of a left eigenvector, by compactness together with primitivity and recurrence.

The starting point of the proofs in the two-letter substitutive case is that strong coincidences hold [4]. We also prove an analogous statement as a starting point. However the fact that strong coincidences hold at order n (i.e., for the product $\sigma_0\sigma_1 \cdots \sigma_n$) does not necessarily imply strong coincidences at order m for $m > n$ (which holds in the substitutive case) makes the extension to the present framework more delicate than it first occurs. The proof heavily relies, among other things, on the recurrence of the directive sequence. There are then two strategies, one based on making explicit the action of the rotation on the unit circle [16], whereas the approach of [17] uses the balanced pair algorithm. However, there seems to be no natural expression of the balanced pair algorithm in the S-adic framework. Our strategy for proving discrete spectrum thus relies on the use of "Rauzy fractals" and follows [16]. We recall that a Rauzy fractal is a set which is endowed with an exchange of pieces acting on it that allows to make explicit (by factorizing by a natural lattice) the