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An uncountable set of tiling spaces with distinct cohomology

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A R T I C L E I N F O

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We generalise the notion of a Barge–Diamond complex, in the one-dimensional case, to any mixed system of tiling substitutions. This gives a way of describing the associated tiling space as an inverse limit of Barge–Diamond complexes. We give an effective method for calculating the Čech cohomology of the tiling space via an exact sequence relating the associated sequence of substitution matrices and certain subcomplexes appearing in the approximants. As an application, we show that there exists a system of three substitutions on two letters which exhibit an uncountable collection of minimal tiling spaces with distinct isomorphism classes of Čech cohomology.

0. Introduction

The dynamical, topological and combinatorial properties of sequences derived from *S-adic systems* have been well studied for many years (see [10] and the references within). They have been classically defined as those sequences which appear as a limit word of a sequence of substitutions $s = \lim_{n\to\infty}((\phi_1 \circ \cdots \circ \phi_n)(a))$ for some letter in a finite alphabet a, and the substitutions ϕ_i belonging to some finite family of substitutions S. In [16], Gähler and Maloney extended this class of sequences to an analogous class of tilings of $\mathbb{R}^n$ which generalise the now well-studied *substitution tilings* – these are referred to as *mixed substitution* or *multisubstitution* tiling systems. It is of general interest to be able to calculate the Čech cohomology and other invariants of moduli spaces of aperiodic tilings, or *tiling spaces* (see [3,4,7,12,13,15,20,21]).

Using techniques developed by authors such as Kellendonk [18], Anderson & Putnam [1], Gähler [14], and Barge & Sadun [5], a general method for calculating the Čech cohomology of tiling spaces associated to mixed substitution systems was developed by Gähler and Maloney. In particular, they showed with an example that the topology of the associated tiling spaces of mixed substitution systems can be dependent on the order in which the substitutions are applied, and not just on the family of substitutions being considered.

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This example provided a set of 1-dimensional mixed substitution tiling spaces over a fixed collection of two substitutions, some of whose first Čech cohomology groups differ, and in fact have ranks varying depending on the choice of the sequence in which the substitutions are applied. This certainly hints that the family of mixed substitution tiling spaces has a richer structure than the classical case of tiling spaces associated to singular substitutions.

One naturally asks how much more wild this class of spaces can behave. We already have some partial results. For instance it is well known that the Sturmian sequences can all be generated as limits of a system of two substitutions on two letters. It is also known that the tiling spaces associated to the Sturmian sequences s_α and s_β are homeomorphic if and only if the generating slopes α and β have continued fraction representations whose tails agree after some finite number of shifts [11,6]. In particular, this tells us that there are an uncountable number of distinct homeomorphism-types of mixed substitution tiling spaces – in contrast with the case of singular substitutions. Unfortunately, these spaces cannot be distinguished by their Čech cohomology which are all isomorphic to the direct sum of two copies of the integers $\mathbb{Z}^2$.

Our main theorem is an improvement on this result. We show, using a theorem of Goodearl and Rushing [17], that an uncountable family exists which can be distinguished by their cohomology.

Theorem 1. *There exists a family of minimal mixed substitution tiling spaces exhibiting an uncountable collection of distinct isomorphism classes of first Čech cohomology groups.*

This has the (perhaps surprising) consequence that there exist tiling spaces Ω for which the first Čech cohomology group $\check{H}^1(\Omega)$ cannot be written in the form

$$A \oplus (\mathbb{Z}[1/n_1] \oplus \cdots \oplus \mathbb{Z}[1/n_k])$$

for finitely generated abelian group A and natural numbers n_i, $1 \leq i \leq k$, because there are only countably many distinct isomorphism classes of such groups. Moreover, it appears that these pathological cohomology groups are in some sense typical. Nevertheless, almost every currently determined cohomology group of a tiling space is of the above form.

In order to prove Theorem 1, we leverage a construction by Barge and Diamond [3] of the so-called BD (Barge–Diamond) complex of a tiling substitution. The BD complex is a CW complex associated to a single tiling substitution, built from combinatorial data and with the property that a suitably chosen continuous map on this complex, induced by the substitution, has an inverse limit which is homeomorphic to the tiling space. A cellular map can then be defined which is homotopic to this induced map, and which also acts simplicially on a particular subcomplex of the BD complex and maps this subcomplex into itself. A relative cohomology approach can then be used to produce an exact sequence which allows for the straight forward computation of the Čech cohomology of the tiling space in terms of the cohomology of a simplicial complex, and the direct limit of the transpose of the transition matrix associated to the substitution. These constructions and results were later generalised to tilings of $\mathbb{R}^n$ for all positive dimensions by Barge, Diamond, Hunton and Sadun in [4].

The original construction of these complexes, and the induced substitution maps between them, was only developed for single substitutions and so it is necessary to generalise their work to the case of mixed substitution systems. We essentially mirror the theory developed by Gähler and Maloney [16] for the Anderson–Putnam complex, but instead for the BD complex, and only in one dimension – though it should be noted that an extension to higher dimensions would not be difficult, given [4].

In Section 1, we introduce notation and basic definitions relating to the 3-adic numbers. We provide an overview of the Goodearl–Rushing result [17] and, to keep the work as self-contained as possible, a reproduction of the proof of this result. Briefly, the result shows that the set of direct limits over $\mathbb{Z}^2$ of arbitrary sequences of matrices of the form $\left(\begin{smallmatrix} 1 & i \\ 0 & 3 \end{smallmatrix}\right)$, $i \in \{0, 1, 2\}$ satisfies the property that only countably