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Finitely balanced sequences and plasticity of 1-dimensional tilings ☆

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ABSTRACT

We relate a balancing property of letters for bi-infinite sequences to the invariance of the resulting 1-dimensional tiling dynamics under changes in the lengths of the tiles. If the language of the sequence space is finitely balanced, then all length changes in the corresponding tiling space result in topological conjugacies, up to an overall rescaling.

1. Introduction and statement of result

Spaces of bi-infinite sequences are closely related to spaces of 1-dimensional tilings. Given a tiling, one can construct a sequence by looking at the labels of the tiles. Given a sequence, one can construct a tiling by associating a tile of length ℓ_i to each letter of type a_i, and by concatenating these tiles in the order given by the sequence.

More precisely, let Ξ be a space of bi-infinite sequences on a finite alphabet $\mathcal{A} = \{a_1, \ldots, a_n\}$, and let $\ell = (\ell_1, \ell_2, \ldots, \ell_n)$ be a vector of positive lengths. We then construct a space Ω_ℓ of tilings obtained by associating each letter a_i with a tile of label i and length ℓ_i. Ω_ℓ admits a natural $\mathbb{R}$ action from translation, while Ξ only admits a natural $\mathbb{Z}$ action obtained from powers of the shift map.

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A natural question is how mixing or spectral properties of the $\mathbb{Z}$ action on Ξ are related to analogous properties of the $\mathbb{R}$ action on Ω_ℓ. When $\ell = (1, \ldots, 1)$, this is a simple question, since the spectrum of translations on Ω_ℓ is simply ($-\sqrt{-1}$ times) the log of the spectrum of the shift operator on Ξ. However, in some tiling spaces changing ℓ can qualitatively change the dynamics, while in others it cannot.

Part of this question was studied in [2,3], and is related to the (Čech) cohomology of Ω_ℓ. Changes in the lengths of the tiles are associated with elements of $H^1(\Omega_\ell, \mathbb{R})$. There is a subgroup $H^1_{an}(\Omega_\ell, \mathbb{R})$ of *asymptotically negligible* length changes, i.e., changes that yield tiling spaces (and $\mathbb{R}$ actions) that are topologically conjugate to the original action on Ω_ℓ. Furthermore, a uniform rescaling of all the tiles, while not a topological conjugacy, does not change the qualitative mixing and spectral properties. If $H^1_{an}(\Omega_\ell, \mathbb{R})$ is a large enough subspace of $H^1(\Omega_\ell, \mathbb{R})$, then all length changes are a combination of a uniform rescaling and a topological conjugacy. This happens, for instance, for homological Pisot substitutions [5].

We call a 1-dimensional tiling space (or the corresponding space of sequences) *plastic* if all changes to lengths of the n basic tiles result in topological conjugacies, up to an overall scale. We call it *totally plastic* if this remains true for all finite recodings of the tiles. (E.g. collaring, or dividing a tile into multiple pieces, or amalgamating tiles.) Plasticity may be a property of how we label our tiles, but total plasticity is topological. A tiling space is totally plastic if and only if $H^1(\Omega_\ell, \mathbb{R})/H^1_{an}(\Omega_\ell, \mathbb{R}) = \mathbb{R}$.

Example 1. The space of Fibonacci sequences is generated by the substitution $\sigma(a) = ab$, $\sigma(b) = a$. The corresponding space of tilings was shown in [12] to be plastic. To see this, suppose that Ω_ℓ and $\Omega_{\ell'}$ are Fibonacci tiling spaces corresponding to length vectors $\ell = (\ell_1, \ell_2)$ and $\ell' = (\ell'_1, \ell'_2)$. By rescaling ℓ or ℓ', we can suppose that $\phi\ell_1 + \ell_2 = \phi\ell'_1 + \ell'_2$, where $\phi = (1 + \sqrt{5})/2$ is the golden mean. This implies that $(\ell_1 - \ell'_1, \ell_2 - \ell'_2)$ is a left-eigenvector of the substitution matrix $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with eigenvalue $1 - \phi$, and hence that the difference in length between n-th order supertiles $\sigma^n(a)$ in Ω_ℓ and $\Omega_{\ell'}$ goes to zero as $n \to \infty$, and likewise for $\sigma^n(b)$. Let $\psi_n : \Omega_\ell \to \Omega_{\ell'}$ be a map the preserves the underlying sequence, and such that if the origin lies a fraction f across an n-th order supertile in a tiling T, then the origin lies the same fraction f across the corresponding supertile in $\psi_n(T)$. The tilings $\psi_n(T)$ all agree up to translation, and the limiting tiling $\psi(T) := \lim_{n\to\infty} \psi_n(T)$ is well-defined. Then ψ is a topological conjugacy. (For a more complete description of this conjugacy, see [12] or [14].)

1.1. Finitely balanced words

Let Ξ be a space of sequences. The set of all finite words that appear within the sequences of Ξ is called the *language* of Ξ, and is denoted $\mathcal{L}$.

For each letter a_i and each integer $n > 0$, let $m_i(n)$ be the minimum number of times that a_i can appear in a word of length n in $\mathcal{L}$, and let $M_i(n)$ be the maximum number of times that a_i can appear. If there is a constant C such that $M_i(n) - m_i(n) < C$ for all i and all n, then we say that the language $\mathcal{L}$ is *finitely balanced.*

Example 2. In a Sturmian sequence, the alphabet $\mathcal{A}$ consists of two letters $a_1 = a$ and $a_2 = b$, and bb is not in the language. We then have $m_1(2) = 1$ and $M_1(2) = 2$, since any word of length 2 contains either one or two a's. In fact, Sturmian sequences have $M_i(n) - m_i(n) = 1$ for all i and n.

We can look for variations in the appearance not only of letters, but of words. Let $w \in \mathcal{L}$, and let $m_w(n)$ (resp. $M_w(n)$) be the minimum (resp. maximum) number of times that w can appear in a word of length n. If for each word w there is a constant C_w such that $M_w(n) - m_w(n) < C_w$ for all n, then we say that $\mathcal{L}$ is *totally finitely balanced.*

Example 3. The space of Thue–Morse sequences, coming from the substitution $\phi(a) = ab$, $\phi(b) = ba$, is finitely balanced but not totally finitely balanced. We have $M_1(n) - m_1(n) \leq 2$ for all n, and likewise