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Topology of the set of univoque bases

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ABSTRACT

Given a positive integer M, a number $q > 1$ is called a *univoque base* if there is exactly one sequence $(c_i) = c_1c_2\cdots$ with integer digits c_i belonging to the set $\{0, 1, \ldots, M\}$, such that $1 = \sum_{i=1}^{\infty} c_i q^{-i}$. The topological and combinatorial properties of the set of univoque bases $\mathcal{U}$ and their corresponding sequences (c_i) have been investigated in many papers since a pioneering work of Erdős, Horváth and Joó 25 years ago. While in most studies the attention was restricted to univoque bases belonging to $(M, M + 1]$, a recent work of Kong and Li on the Hausdorff dimension of unique expansions demonstrated the necessity to extend the earlier results to all univoque bases. This is the object of this paper. Although the general research strategy remains the same, a number of new arguments are needed, several new properties are uncovered, and some formerly known results become simpler and more natural in the present framework.

1. Introduction

Fix a positive integer M and an *alphabet* $\{0, 1, \ldots, M\}$. By a *sequence* we mean an element $c = (c_i) = c_1c_2\cdots$ of $\{0, 1, \ldots, M\}^\infty$. We will frequently use the lexicographic order between sequences and blocks (i.e., elements of $\{0, 1, \ldots, M\}^n$ for some $n \geq 1$). Furthermore, we give each coordinate $\{0, 1, \ldots, M\}$ the discrete topology and endow the set $\{0, 1, \ldots, M\}^\infty$ with the Tychonoff product topology. The corresponding convergence is the component-wise convergence of sequences.

Given a real *base* $q > 1$, by an *expansion* of a real number x we mean a sequence $c = (c_i)$ satisfying the equality

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$$\sum_{i=1}^{\infty} \frac{c_i}{q^i} = x.$$

Expansions of this type in *non-integer* bases have been extensively investigated since a pioneering paper of Rényi [22].

One of the striking features of non-integer bases is that if $q \in (M, M+1)$, almost every number in $[0, M/(q-1)]$ has a continuum of expansions, see, e.g., [23]. In fact, using the theory of *random β-expansions* (see [4]) it was shown by [3] that almost every $x \in [0, M/(q-1)]$ has a continuum of so called *universal* expansions, i.e., expansions containing all possible blocks in $\{0, 1, \ldots, M\}^n$, $n = 1, 2, \ldots$. On the other hand, [10] constructed numbers $q \in (M, M+1)$ such that $x = 1$ has a unique expansion in base q, a discovery that stimulated many works during the past 25 years. We refer to the surveys [24,13,9] for more information.

In this paper we investigate only expansions of $x = 1$. (The only exception is Theorem 2.1 and its applications in the proof of several lemmas.) Hence by a *q-expansion* or an *expansion* we mean a sequence $c = (c_i)$ satisfying the equality

$$\pi_q(c) := \sum_{i=1}^{\infty} \frac{c_i}{q^i} = 1. \tag{1.1}$$

Since

$$0 \le \sum_{i=1}^{\infty} \frac{c_i}{q^i} \le \frac{M}{q-1}$$

for all sequences (c_i), such expansions may exist only if $q \in (1, M+1]$. Conversely, the greedy algorithm of [22] provides a q-expansion for each $q \in (1, M+1]$, which is defined recursively as follows: if for some positive integer n the digits $\beta_1, \ldots, \beta_{n-1}$ are already defined (no condition if $n = 1$), then β_n is the largest digit in $\{0, \ldots, M\}$ such that the inequality $\sum_{i=1}^{n} \beta_i q^{-i} \le 1$ holds. The resulting *greedy* or *β-expansion* $\beta(q)$ or $(\beta_i(q))$ of $x = 1$ is clearly the lexicographically largest q-expansion.

Examples 1.1. Let $M = 1$.

(i) If $q \in (1, \varphi)$, where $\varphi \approx 1.618$ denotes the Golden Ratio, then there is a continuum of q-expansions [11].
(ii) If $q = \varphi$, then there are countably many q-expansions [10]: symbolically

$$(10)^\infty, \quad \text{and} \quad (10)^k 110^\infty, \quad (10)^k 01^\infty, \quad k = 0, 1, \ldots.$$

(iii) If $q = \varphi_n$ is a Multinacci number, i.e., the positive solution of $q^n = q^{n-1} + \cdots + q + 1$ for some $n = 3, 4, \ldots$, then there are countably many q-expansions [18]:

$$(1^{n-1}0)^\infty, \quad \text{and} \quad (1^{n-1}0)^k 1^n 0^\infty, \quad k = 0, 1, \ldots.$$

(iv) If $q = 2$, then 1^∞ is the only q-expansion.
(v) Let $c = 1(1^n 0^k)^\infty$ for some integers $n \ge k \ge 1$, and define $q \in (1, 2)$ by the equation (1.1). Then c is the only q-expansion [11,10].

See also [13] for short elementary proofs of the above statements.

A sequence is called *finite* if it has a last non-zero digit, and *infinite* otherwise. Thus 0^∞ is considered to be an infinite sequence: this unusual terminology simplifies many statements in the sequel.