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A note on the planar surface sum of Heegaard splittings [☆]

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ABSTRACT

In this paper, we prove that under some condition, (1) the planar surface sum of two unstabilized Heegaard splittings is unstabilized; (2) the self planar surface sum of an unstabilized Heegaard splitting is unstabilized.

1. Introduction

All manifolds are assumed to be compact and orientable, unless stated. When M is a 3-manifold with $\partial M \neq \emptyset$ and C is the union of finitely many disjoint simple closed curves on ∂M, we will always denote $M[C]$ the 3-manifold obtained by adding 2-handles to M along C and capping off possible resulting 2-sphere boundary components with 3-balls.

A compression body V is a 3-manifold which can be obtained by attaching some number of 2-handles to $F \times [0, 1]$ along finitely many simple closed curves in $F \times \{1\}$, and capping off each resulting 2-sphere boundary component with a 3-ball, where F is a closed surface. $F \times \{0\}$ is denoted by $\partial_+ V$, and $\partial V - \partial_+ V$ is denoted by $\partial_- V$. If there is a closed surface S which cuts M into two compression bodies V and W such that $S = \partial_+ V = \partial_+ W$, then we say M has a Heegaard splitting $V \cup_S W$. The genus of S is called the genus of the Heegaard splitting $V \cup_S W$. The Heegaard genus of M, denoted by $g(M)$ is the minimal genera of all Heegaard splittings of M.

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When M is a Haken manifold, i.e., M is irreducible and contains an incompressible surface F, then M can be obtained by amalgamating two manifolds if F is separating in M, or by self amalgamating a connected manifold if F is non-separating. When F is closed, there are many results about unstabilized and minimal genus Heegaard splittings of amalgamations over F, see Lackenby [4], Souto [9], Li [5], Kobayashi, Qiu [3], Yang, Lei [10], Du, Qiu [2], Zou, Du, Guo, Qiu [11], etc. When F has non-empty boundary, there are only a few results, see Qiu, Wang, Zhang [8]. In this paper, we consider the case that F is a connected essential planar surface in M. Since Ma and Qiu [6] studied the amalgamation and self-amalgamation along F when F is an essential disk, we assume that $\chi(F) \leq 0$ here. In this paper, we prove following results.

Theorem 1.1. *Let F be an essential planar surface in a 3-manifold M such that F separates M into two manifolds M_1 and M_2. Let F_i be the copy of F in M_i and suppose $V_i \cup_{S_i} W_i$ is a Heegaard splitting of M_i such that $F_i \subset \partial_- W_i$ (for $i = 1, 2$). Suppose that $V_1 \cup_{S_1} (W_1[\partial F_1])$ and $V_2 \cup_{S_2} (W_2[\partial F_2])$ are unstabilized, then the planar surface sum (defined in Section 2) of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ is an unstabilized Heegaard splitting of M.*

Remark 1.1. In Theorem 1.1, $W_1[\partial F_1]$ is a compression body with $\partial_+(W_1[\partial F_1]) = \partial_+ W_1 = S_1$ since $\partial F_1 \subset \partial_- W_1$, hence $V_1 \cup_{S_1} (W_1[\partial F_1])$ is a Heegaard splitting of $M_1[\partial F_1]$. For the same reason, $W_2[\partial F_2]$ is a compression body and $V_2 \cup_{S_2} (W_2[\partial F_2])$ is a Heegaard splitting of $M_2[\partial F_2]$.

Theorem 1.2. *Let F be a non-separating essential planar surface in a 3-manifold M, and let M_1 be the manifold obtained by cutting M along F. Denote by F_1 and F_2 the two copies of $F \subset M_1$ after this cutting. Let $V_1 \cup_{S_1} W_1$ be a Heegaard splitting of M_1 such that $F_1 \cup F_2 \subset \partial_- W_1$. Suppose that $V_1 \cup_{S_1} (W_1[\partial F_1 \cup \partial F_2])$ is unstabilized, then the self planar surface sum (defined in Section 2) of $V_1 \cup_{S_1} W_1$ is an unstabilized Heegaard splitting of M.*

Remark 1.2. In Theorem 1.2, we can also see $W_1([\partial F_1 \cup \partial F_2])$ is a compression body with $\partial_+(W_1[\partial F_1 \cup \partial F_2]) = \partial_+ W_1 = S_1$, since $F_1 \cup F_2 \subset \partial_- W_1$. Hence we also have a Heegaard splitting $M_1[\partial F_1 \cup \partial F_2] = V_1 \cup_{S_1} W_1([\partial F_1 \cup \partial F_2])$.

2. Preliminaries

A Heegaard splitting $M = V \cup_S W$ is *stabilized* if there are essential disks $D \subset V$ and $E \subset W$ such that ∂D intersects ∂E in one point. Otherwise, the Heegaard splitting $V \cup_S W$ is *unstabilized*. Suppose there is a separating sphere F in $M = V \cup_S W$ such that $F \cap S$ is a circle, and suppose that $F \cap V = D_1$ and $F \cap W = D_2$. Then D_1 separates V into two compression bodies V_1' and V_2', and D_2 separates W into two compression bodies W_1 and W_2. Assume that V_i' and W_i are in the same side of F. Let $M_i' = V_i' \cup W_i$, then M_i' is a connected manifold and contains a sphere boundary component F_i which is a copy of F after cutting M along F. Let B_i be a 3-ball, and let $M_i = M_i' \cup_{F_i = \partial B_i} B_i$ and $V_i = V_i' \cup B_i \subset M_i$. Then $V_i \cup W_i$ is a Heegaard splitting of M_i. We say that the Heegaard splitting $V \cup W$ is the *connected sum* of these two Heegaard splittings of $V_1 \cup W_1$ and $V_2 \cup W_2$.

Following is a conjecture about stability of connected sum of Heegaard splittings proved by Bachman [1] and Qiu [7] independently.

Theorem 2.1 *(Gordon conjecture). The connect sum of any two unstabilized Heegaard splittings is unstabilized.*

Next we introduce (planar) surface sum and self (planar) surface sum of Heegaard splitting(s) as in [8], which are central notions in this paper.