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Big fundamental groups: Generalizing homotopy and big homotopy

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The concept of big homotopy theory was introduced by J. Cannon and G. Conner in [1] using big intervals of arbitrarily large cardinality to detect big loops. We find, for each space, a canonical cardinal that is sufficient to detect all big loops and all big homotopies in the space.

1. Introduction

Cannon and Conner [1] defined big homotopy theory and proved that for any Hausdorff space, the big fundamental group is well-defined. In this paper, we show that it is well-defined for any pointed space X and calculate a canonical cardinal $\gamma(X)$ that is sufficient for computing the big fundamental group. In particular, for any cardinal α, we define the functor π_1^α using big intervals of cardinality at most α, show that for any X, the big fundamental group $\Pi_1(X)$ is naturally realized by π_1^α, and even find a canonical interval $I(X)$ such that any big loop in X can be realized as a path from $I(X)$.

We state here some of the definitions and results from that paper as a convenience to the reader and to provide background and motivation for the main part of the paper, which is to improve upon the theory and generate a family of functors to calculate homotopy in a slightly different way.

The concept of big homotopy theory is simply to extend the idea of a loop to allow for objects other than the real interval $[0, 1]$ to be the domain of a loop. More specifically, a *big interval* is a totally-ordered

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set which is compact and connected. Equivalently, a totally-ordered set which is order complete, dense, and has a maximal and minimal element. A big interval is endowed with the order topology. Then a big path is merely a continuous function whose domain is a big interval. A big loop is a big path whose endpoints coincide. For notational purposes, given a big interval I, the minimal element will be denoted 0_I and 1_I will denote its maximal element.

Big path homotopy is slightly more complicated to define. Instead of simply a function which starts at one path and ends at the other, we require a slightly weaker criterion. The big paths $f : I \to X$ and $g : J \to X$ are said to be *big path homotopic* if the there are big intervals A and B, a continuous function $H : A \times B \to X$, and surjective order-preserving maps $p : A \to I$ and $q : B \to J$ such that the following are satisfied.

1. $f(0_I) = g(0_J)$ and $f(1_I) = g(1_J)$ that is, f and g agree on their endpoints.
2. $H(s, 0_B) = f(p(s))$ for all $s \in A$.
3. $H(s, 1_B) = g(q(s))$ for all $s \in A$.
4. $H(0_A, t)$ and $H(1_A, t)$ are constant for all $t \in B$.

When such functions exist, we say that the triple (H, p, q) forms a big homotopy from f to g. We think of B as the parameter space and when this space is significant, we call the triple a *B-homotopy.* Big path homotopy is shown to be an equivalence relation. Given a pointed topological space (X, x_0), the class of big loops based at x_0 reduces to a set of equivalence classes of loops using this definition. In [1], it is shown that this is a set when X is Hausdorff. A slight modification of the proof shows that it is true for all X. This set is called the *big fundamental group* of X based at x_0 and is denoted by $\Pi_1(X, x_0)$. This set is turned into a group by defining path concatenation as the group operation and reverse path as the inverse operation. In this paper, we define for each cardinal α a new group called π_1^α and show that the big fundamental group is realized by π_1^α, for some α depending on X. In [2], Turner shows that many of the same results from classical homotopy theory carry over, such as covering space theory.

2. Big intervals

One primary purpose of the paper is to construct, given a cardinal α, a functor π_1^α which will give a big fundamental group given a pointed space (X, x_0). For now we will define $\pi_1^\alpha(X, x_0)$ using all big loops in X based at x_0 whose domain has density at most α, and big homotopies also with parameter spaces of density at most α. We will see below that it is possible to construct one big interval (depending on α) which by itself will allow for all the big paths and big homotopies allowed by this definition.

Proposition 2.1. *Let I and J be big intervals. The following are equivalent.*

1. *There is an order-preserving injection $f : I \to J$*
2. *There is a dense subset $A \subset I$ and an order-preserving injection $h : A \to J$.*
3. *There is an order-preserving surjection $g : J \to I$*

Proof. Condition (2) follows immediately from condition (1). We show that (2) implies (3), so assume h exists. Define g as follows. For a point $j \in J$, define the set $S_j = \{i \in A \mid h(i) \leq j\}$. Then define $g(j) = \sup S_j$. Note that for any j such that $S_j = \emptyset$, we have $g(j) = 0_I$. The property of order-preserving is obvious. To see that it is surjective, let $i \in I$. Let $j = \sup\{h(i') \mid i' < i,\ i' \in A\}$. Since A is dense in I, we see that $i = \sup S_j$ because for any $i' < i$, it must be that $h(i') < j$. Thus g is surjective.