Topology and its Applications 204 (2016) 41–50

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Strongly proximal continuity & strong connectedness

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ARTICLE INFO

Article history:
Received 28 April 2015
Accepted 15 February 2016
Available online 27 February 2016

MSC:
54D05
54C08
54E05
54B20

Keywords:
Connected
Hypertopology
Strongly proximally continuous
Strong proximal equivalence
Strongly proximally connected

ABSTRACT

This article introduces strongly proximal continuous (s.p.c.) functions, strong proximal equivalence (s.p.e.) and strong connectedness. A main result is that if topological spaces X, Y are endowed with compatible strong proximities and $f : X \longrightarrow Y$ is a bijective s.p.e., then its extension on the hyperspaces $\mathrm{CL}(X)$ and $\mathrm{CL}(Y)$, endowed with the related strongly hit and miss hypertopologies, is a homeomorphism. For a topological space endowed with a strongly near proximity, strongly proximal connectedness implies connectedness but not conversely. Conditions required for strongly proximal connectedness are given. Applications of s.p.c. and strongly proximal connectedness are given in terms of strongly proximal descriptive proximity.

1. Introduction

This article carries forward recent work on strong proximity [27,26,24,25] and strongly hit and miss hypertopologies [28]. Strongly proximal continuous functions and strongly near connectedness of subsets in topological spaces are introduced.

2. Preliminaries

Proximities are a powerful tool in dealing with the concept of nearness without involving metrics (see, *e.g.*, [5,19,6,7]). Proximities are binary relations on the power set $\mathscr{P}(X)$ of a nonempty set X. $A\ \delta\ B$ reads

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[1] The research has been supported by the Natural Sciences & Engineering Research Council of Canada (NSERC) discovery grant 185986.

http://dx.doi.org/10.1016/j.topol.2016.02.008

A is near B. From the usual proximity space axioms, it suffices to have $A \cap B \neq \emptyset$ to obtain $A\ \delta\ B$. We require something more. We want to talk about a stronger kind of nearness. For this reason we introduced *strong proximities* in [27]. Strong proximities satisfy the following axioms.

Definition 1. Let X be a topological space, $A, B, C \subset X$ and $x \in X$. The relation $\overset{\wedge\wedge}{\delta}$ on $\mathscr{P}(X)$ is a *strong proximity*, provided it satisfies the following axioms.

(N0) $\emptyset \overset{\wedge\wedge}{\not\delta} A, \forall A \subset X$, and $X \overset{\wedge\wedge}{\delta} A, \forall A \subset X$
(N1) $A \overset{\wedge\wedge}{\delta} B \Leftrightarrow B \overset{\wedge\wedge}{\delta} A$
(N2) $A \overset{\wedge\wedge}{\delta} B \Rightarrow A \cap B \neq \emptyset$
(N3) If $\{B_i\}_{i \in I}$ is an arbitrary family of subsets of X and $A \overset{\wedge\wedge}{\delta} B_{i^*}$ for some $i^* \in I$ such that $\text{int}(B_{i^*}) \neq \emptyset$, then $A \overset{\wedge\wedge}{\delta} (\bigcup_{i \in I} B_i)$
(N4) $\text{int}A \cap \text{int}B \neq \emptyset \Rightarrow A \overset{\wedge\wedge}{\delta} B$ $\square$

When we write $A \overset{\wedge\wedge}{\delta} B$, we read A is *strongly near* B. For each *strong proximity*, we assume the following relations:

N5) $x \in \text{int}(A) \Rightarrow x \overset{\wedge\wedge}{\delta} A$
N6) $\{x\} \overset{\wedge\wedge}{\delta} \{y\} \Leftrightarrow x = y$ $\square$

So, for example, if we take the strong proximity related to non-empty intersection of interiors, we have that $A \overset{\wedge\wedge}{\delta} B \Leftrightarrow \text{int}A \cap \text{int}B \neq \emptyset$ or either A or B is equal to X, provided A and B are not singletons; if $A = \{x\}$, then $x \in \text{int}(B)$, and if B too is a singleton, then $x = y$. It turns out that if $A \subset X$ is an open set, then each point that belongs to A is strongly near A.

Related to this new kind of nearness introduced in [27] which extends traditional proximity (see, *e.g.*, [2, 17,14–16,20,29,12]), we defined a new kind of *hit-and-miss hypertopology* [27,28], which extends recent work on hypertopologies (see, *e.g.*, [1,3,4,8–11,13,18]). The important thing to notice that this work has its foundation in geometry [13,23,24].

The *strongly hit and far-miss* topology $\tau^{\wedge\wedge}$ has as subbase the sets of the form:

- $V^{\wedge\wedge} = \{E \in \text{CL}(X) : E \overset{\wedge\wedge}{\delta} V\}$, where V is an open subset of X,
- $A^{++} = \{\ E \in \text{CL}(X) : E \not\delta X \setminus A\ \}$, where A is an open subset of X.

In [28], we considered the Hausdorffness of the previous topology associated with suitable families of subsets.

In this paper we go deeper into the study of *strong proximities* in terms of concepts of strong proximal continuity and strong proximal connectedness. Moreover, these new forms of proximal continuity and connectedness are applied in some examples of descriptive nearness, which is particularly useful for many applications.

3. Strongly proximal continuity

After introducing strong proximities, the natural continuation is to look at mappings that preserve proximal structures. We call such *strongly proximal continuous* mappings.

Definition 2. Suppose that $(X, \tau_X, \overset{\wedge\wedge}{\delta}_X)$ and $(Y, \tau_Y, \overset{\wedge\wedge}{\delta}_Y)$ are topological spaces endowed with strong proximities.