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Additivity of the ideal of microscopic sets

Adam Kwela

Institute of Mathematics, University of Gdańsk, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland

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ABSTRACT

A set $M \subset \mathbb{R}$ is microscopic if for each $\varepsilon > 0$ there is a sequence of intervals $(J_n)_{n\in\omega}$ covering M and such that $|J_n| \leq \varepsilon^{n+1}$ for each $n \in \omega$. We show that there is a microscopic set which cannot be covered by a sequence $(J_n)_{n\in\omega}$ with $\{n \in \omega : J_n \neq \emptyset\}$ of lower asymptotic density zero. We prove (in ZFC) that additivity of the ideal of microscopic sets is ω_1. This solves a problem of G. Horbaczewska. Finally, we discuss additivity of some generalizations of this ideal.

1. Introduction

For $n \in \omega$ we use the identification $n = \{0, 1, \dots, n-1\}$. By $\mathrm{card}(A)$ we denote cardinality of a set A. For an interval $I \subset \mathbb{R}$ by $|I|$ we denote its length. Given $r \in \mathbb{R}$ and $A \subset \mathbb{R}$ we write $r \cdot A = \{ra : a \in A\}$ and $r + A = \{r + a : a \in A\}$.

We say that a sequence of intervals $(J_n)_{n\in\omega}$ *covers the set* $A \subset \mathbb{R}$ if $A \subset \bigcup_{n\in\omega} J_n$.

Definition 1.1. (J. Appell [1]) A set $M \subset \mathbb{R}$ is called *microscopic* if for each $\varepsilon > 0$ there exists a sequence of intervals $(J_n)_{n\in\omega}$ covering M and such that $|J_n| \leq \varepsilon^{n+1}$ for each $n \in \omega$. The family of all microscopic sets will be denoted by $\mathcal{M}$.

This notion was introduced in 2001 by J. Appell in [1]. Deeper studies of microscopic sets were done by J. Appell, E. D'Aniello and M. Väth in [2]. Since that time, several papers were devoted to this subject, including [9,10] and [11]. In [8] one can find a summary of the progress made in this area.

It is easy to see that every microscopic set is contained in some microscopic $\mathbf{G}_\delta$-set, i.e., $\mathcal{M}$ is $\mathbf{G}_\delta$-generated (cf. [8, Theorem 1.1]). Moreover, $\mathcal{M}$ is strictly smaller than the σ-ideal of sets of Lebesgue measure zero (cf. [8]). Therefore, many classical theorems stating that some property holds everywhere except a set of Lebesgue measure zero, are being strengthened by showing that actually the set of exceptions can be chosen

E-mail address: Adam.Kwela@ug.edu.pl.

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to be microscopic. For instance, it can be proved that $\mathbb{R}$ can be decomposed into two sets such that one of them is of first category and the second one is microscopic (cf. [9]).

The aim of this paper is to determine the smallest number of sets from $\mathcal{M}$ union of which is not in $\mathcal{M}$ anymore. For this purpose, we need the notion of asymptotic density of a subset of ω.

Recall that for any $A \subset \omega$ its *upper and lower asymptotic density* are given by the formulas:

$$\overline{d}(A) = \limsup_{j\to\infty} \frac{\mathrm{card}(A \cap (j+1))}{j+1},$$
$$\underline{d}(A) = \liminf_{j\to\infty} \frac{\mathrm{card}(A \cap (j+1))}{j+1}.$$

If $\overline{d}(A) = \underline{d}(A)$, then we say that the set A *is of asymptotic density* $d(A)$ which is equal to this common value.

Definition 1.2. Let $\delta \in [0,1]$. We say that a microscopic set $M \subset \mathbb{R}$ *admits a cover of (lower) asymptotic density* δ if for every $\varepsilon > 0$ there is $D \subset \omega$ with $d(D) \leq \delta$ ($\underline{d}(D) \leq \delta$) and a sequence of intervals $(J_d)_{d\in D}$ which covers M and satisfies $|J_d| \leq \varepsilon^{d+1}$ for all $d \in D$.

Remark 1.3. It is easy to see that any microscopic set $M \subset \mathbb{R}$ admits a cover of arbitrarily small positive asymptotic density. Actually, for any $k \in \omega$ and $\varepsilon > 0$ one can find a sequence of intervals $(J_d)_{d\in D}$, where $D = (k+1)\cdot(\omega+1)$, which covers M and satisfies $|J_d| \leq \varepsilon^{d+1}$ for each $d \in D$.

Indeed, set any $k \in \omega$ and $\varepsilon > 0$. Since M is microscopic, there is a sequence of intervals $(J'_n)_{n\in\omega}$ covering M with $|J'_n| \leq \left(\varepsilon^{k+1}\right)^{n+1} = \varepsilon^{(k+1)(n+1)}$ for each $n \in \omega$. Then it suffices to put $J_{(k+1)(n+1)} = J'_n$ for $n \in \omega$.

In Section 3 we will show that the above cannot be strengthened, i.e., there is a microscopic set which does not admit a cover of lower asymptotic density zero (cf. Theorem 3.1).

From Remark 1.3 it easily follows that $\mathcal{M}$ is a σ-ideal (see [2] or [8] for details). Among studies of σ-ideals, examination of cardinal coefficients related to them has been of great interest during last decades. This is due to the famous Cichoń's diagram which classifies cardinal coefficients of the ideals of null sets and meager sets (cf. [3] and [6]).

Recall the definitions of *additivity, covering number, uniformity number and cofinality of an ideal* $\mathcal{I}$ of subsets of $\mathbb{R}$:

$$\mathtt{add}\,(\mathcal{I}) = \min\left\{\mathrm{card}(\mathcal{A}) : \quad \mathcal{A} \subset \mathcal{I} \quad \wedge \quad \bigcup \mathcal{A} \notin \mathcal{I}\right\};$$
$$\mathtt{cov}\,(\mathcal{I}) = \min\left\{\mathrm{card}(\mathcal{A}) : \quad \mathcal{A} \subset \mathcal{I} \quad \wedge \quad \bigcup \mathcal{A} = \mathbb{R}\right\};$$
$$\mathtt{non}\,(\mathcal{I}) = \min\left\{\mathrm{card}(A) : \quad A \subset \mathbb{R} \quad \wedge \quad A \notin \mathcal{I}\right\};$$
$$\mathtt{cof}\,(\mathcal{I}) = \min\left\{\mathrm{card}(\mathcal{B}) : \quad \mathcal{B} \subset \mathcal{I} \quad \wedge \quad \forall_{A\in\mathcal{I}}\exists_{B\in\mathcal{B}} A \subset B\right\}.$$

One can easily prove the following inequalities:

$$\mathtt{add}(\mathcal{I}) \leq \mathtt{non}(\mathcal{I}) \leq \mathtt{cof}(\mathcal{I}) \quad \text{and} \quad \mathtt{add}(\mathcal{I}) \leq \mathtt{cov}(\mathcal{I}) \leq \mathtt{cof}(\mathcal{I}).$$

For more on cardinal coefficients see e.g. [3] or [6].

For the ideal of microscopic sets each of those cardinal coefficients lies between ω_1 and 2^ω (possibly is equal to one of those two numbers), since $\mathcal{M}$ is a σ-ideal of subsets of $\mathbb{R}$ containing all singletons and $\mathbf{G}_\delta$-generated. The aim of this paper is to determine additivity of the ideal of microscopic sets. This problem