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Notes on star Lindelöf space ☆

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ABSTRACT

In this paper, we prove that the cardinality of a star Lindelöf space X does not exceed $\mathfrak{c}$ if X satisfies one of the following conditions: (1) X has a rank 3-diagonal; (2) X is normal and has a rank 2-diagonal; (3) X is first countable, normal and has a G_δ-diagonal. Moreover, we also obtain several results concerning the general question "When must a star Lindelöf space be star countable?".

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1. Introduction

In this paper, we prove that the cardinality of a star Lindelöf space X does not exceed $\mathfrak{c}$ if X satisfies one of the following conditions: (1) X has a rank 3-diagonal; (2) X is normal and has a rank 2-diagonal; (3) X is first countable, normal and has a G_δ-diagonal. Moreover, we also obtain several results concerning the general question "When must a star Lindelöf space be star countable?".

All spaces are assumed to be Hausdorff unless otherwise stated. The cardinality of a set X is denoted by $|X|$, and $[X]^2$ will denote the set of two-element subsets of X. We write ω for the first infinite cardinal and $\mathfrak{c}$ for the cardinality of the continuum.

If A is a subset of X and $\mathcal{U}$ is a family of subsets of X, then $\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. We also put $\mathrm{St}^0(A,\mathcal{U}) = A$ and for negative integer n, $\mathrm{St}^{n+1}(A,\mathcal{U}) = \mathrm{St}(\mathrm{St}^n(A,\mathcal{U}),\mathcal{U})$. If $A = \{x\}$ for some $x \in X$, then we write $\mathrm{St}^n(x,\mathcal{U})$ instead of $\mathrm{St}^n(\{x\},\mathcal{U})$ for simplicity.

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Definition 1.1. Let $\mathcal{P}$ be a topological property. A topological space X is said to be star $\mathcal{P}$, if for any open cover $\mathcal{U}$ of X there is a subset $A \subset X$ with property $\mathcal{P}$ such that $\mathrm{St}(A, \mathcal{U}) = X$. The set A will be called a star kennel of the cover $\mathcal{U}$.

Therefore, a topological space X is said to be star Lindelöf, if for any open cover $\mathcal{U}$ of X there is a Lindelöf subspace $A \subset X$ such that $\mathrm{St}(A, \mathcal{U}) = X$.

Definition 1.2. ([1]) A diagonal sequence of rank k on a space X, where $k \in \omega$, is a countable family $\{\mathcal{U}_n : n \in \omega\}$ of open coverings of X such that $\{x\} = \bigcap\{\mathrm{St}^k(x, \mathcal{U}_n) : n \in \omega\}$ for each $x \in X$.

Definition 1.3. ([1]) A space X has a rank k-diagonal, where $k \in \omega$, if there is a diagonal sequence $\{\mathcal{U}_n : n \in \omega\}$ on X of rank k.

We say that a topological space X has a G_δ-diagonal if there exists a sequence of $\{G_n : n \in \omega\}$ of open sets in X^2 such that $\Delta_X = \bigcap\{G_n : n < \omega\}$, where $\Delta_X = \{(x, x) : x \in X\}$. A space X has a G_δ-diagonal if and only if X has a rank 1-diagonal [11].

Definition 1.4. A space X is said to be metaLindelöf if every open cover of X has a point-countable open refinement.

All notations and terminology not explained here is given in [3].

2. Cardinal inequalities

We will use the following countable version of a set-theoretic theorem due to Erdös and Radó.

Lemma 2.1. *([6, p. 8]) Let X be a set with $|X| > \mathfrak{c}$ and suppose $[X]^2 = \bigcup\{P_n : n \in \omega\}$. Then there exists $n_0 < \omega$ and a subset S of X with $|S| > \omega$ such that $[S]^2 \subset P_{n_0}$.*

Lemma 2.2. *Let $\{\mathcal{U}_n : n \in \omega\}$ be a diagonal sequence on X of rank k, where $k \geq 1$. If $|X| > \mathfrak{c}$, then there exists an uncountable closed discrete subset S of X such that for any two distinct points $x, y \in S$ there exists $n_0 \in \omega$ such that $y \notin \mathrm{St}^k(x, \mathcal{U}_{n_0})$.*

Proof. By our assuming, there exists a sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X such that $\{x\} = \bigcap\{\mathrm{St}^k(x, \mathcal{U}_n) : n \in \omega\}$ for every $x \in X$. We may suppose $\mathrm{St}^k(x, \mathcal{U}_{n+1}) \subset \mathrm{St}^k(x, \mathcal{U}_n)$ for any $n \in \omega$. For each $n \in \omega$ let

$$P_n = \left\{\{x, y\} \in [X]^2 : x \notin \mathrm{St}^k(y, \mathcal{U}_n)\}\right\}.$$

Thus, $[X]^2 = \bigcup\{P_n : n \in \omega\}$. Then by Lemma 2.1 there exists a subset S of X with $|S| > \omega$ and $[S]^2 \subset P_{n_0}$ for some $n_0 \in \omega$. It is evident that for any two distinct points $x, y \in S$, $y \notin \mathrm{St}^k(x, \mathcal{U}_{n_0})$. Now we show that S is closed and discrete. If not, let $x \in X$ and suppose x were an accumulation point of S. Since X is T_1, each neighborhood $U \in \mathcal{U}_{n_0}$ of x meets infinitely many members of S. Therefore there exist distinct points y and z in $S \cap U$. Thus $y \in U \subset \mathrm{St}(z, \mathcal{U}_{n_0}) \subset \mathrm{St}^k(z, \mathcal{U}_{n_0})$. It is a contradiction. Thus S has no accumulation points in X; equivalently, S is a closed and discrete subset of X. This completes the proof. □

Remark 2.3. In the Lemma 2.2, if the diagonal rank of X is at least 2, i.e., $k \geq 2$, then S has a disjoint open expansion $\{\mathrm{St}(x, \mathcal{U}_{n_0}) : x \in S\}$. Indeed, if there exist $x, y \in S$ such that $\mathrm{St}(x, \mathcal{U}_{n_0}) \cap \mathrm{St}(y, \mathcal{U}_{n_0}) \neq \emptyset$, and hence $y \in \mathrm{St}^2(x, \mathcal{U}_{n_0}) \subset \mathrm{St}^k(x, \mathcal{U}_{n_0})$. This is impossible.

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