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On fixed points of locally and pointwise contracting maps

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ABSTRACT

We prove that the following self-mappings must have unique fixed points: *pointwise contractive* (PC) maps on compact rectifiably path connected spaces; *uniformly locally contractive* (ULC) maps on complete connected spaces.

1. Introduction

Let $\langle X, d\rangle$ be a complete metric space. A mapping $f: X \to X$ is *contractive* (*with a contraction constant* λ), abbreviated (C), provided there exists a $\lambda \in [0,1)$ such that $d(f(x), f(y)) \leq \lambda d(x,y)$ for every $x, y \in X$. The Banach Fixed Point Theorem also known as Contraction Mapping Principle [1] states the following:

Theorem 1.1. *(Banach 1922 [1]) If X is a complete metric space and $f: X \to X$ is contractive, then f has a unique fixed point, that is, there exists a unique $\xi \in X$ such that $f(\xi) = \xi$.*

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This Banach result inspired many generalizations, among which we are the most interested in those, where the assumption of contractiveness is relaxed to a local condition, see [3,5,6,10,11,17], and the survey [16]. This work was the most influenced by the 1978 theorem of Hu and Kirk [10], see below, with a proof corrected, in 1982, by Jungck [11].

Definition 1.2. A map $f\colon X \to X$ is *pointwise contractive*, denoted (PC), if for every point $x \in X$ there exists a $\lambda_x \in [0,1)$ and an open neighborhood $U_x \subseteq X$ of x such that $d(f(x), f(y)) \le \lambda_x d(x,y)$ for all $y \in U_x$.[1] We say that f is *uniformly pointwise contractive*, denoted (uPC),[2] provided the same $\lambda \in [0,1)$ works for all $x \in X$.

Theorem 1.3. *(Hu and Kirk 1978 [10]; Jungck 1982 [11]) If $\langle X, d\rangle$ is a rectifiably path connected complete metric space and a map $f\colon X \to X$ is* (uPC)*, then f has a unique fixed point.*

Recall, that a metric space $\langle X, d\rangle$ is *rectifiably path connected* provided any two points $x, y \in X$ can be connected in X by a path $p\colon [0,1] \to X$ of finite length $\ell(p)$, that is, by a continuous map p satisfying $p(0) = x$ and $p(1) = y$, and having a finite length $\ell(p)$ defined as the supremum over all numbers: $\sum_{i=1}^{n} d(p(t_i), p(t_{i-1}))$, where $0 < n < \omega$ and $0 = t_0 < t_1 < \cdots < t_n = 1$.

It is worth noting that Munkres [14, p. 182] provides an example of a (PC) map $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{2}\left(x + \sqrt{x^2+1}\right)$, without fixed or even periodic points. This shows that in Theorem 1.3 the assumption of (uPC) cannot be weakened to (PC). One of two principal results of this paper is to show, that the fixed point result with the weaker requirement of f being (PC) remains true, when we additionally assume that X is compact.

Theorem 1.4. *Assume that $\langle X, d\rangle$ is compact and rectifiably path connected metric space. If $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is* (PC)*, then f has a unique fixed point.*

Theorem 1.4 stands in contrast with the main result of [2, Theorem 1], where we construct a (uPC)[3] self-map on a compact (zero-dimensional) subset X of $\mathbb{R}$ with every orbit being dense, hence having neither fixed nor periodic points.

Our second result on fixed points, without the assumption that the space is rectifiably path connected, requires the following natural definition, in which $B(x, \varepsilon)$ denotes an open ball centered at x and of radius ε.

Definition 1.5. A map $f\colon X \to X$ is *locally contractive*, denoted (LC), if for every $x \in X$ there exist numbers $\lambda_x \in [0,1)$ and $\varepsilon_x > 0$ such that $f \restriction B(x, \varepsilon_x)$ is contractive with constant λ_x. Moreover, f is *uniformly locally contractive* (ULC), if the same λ and ε work for all $x \in X$, which we also indicate by saying that f is (ε, λ)-(ULC).

Theorem 1.6. *Assume that $\langle X, d\rangle$ is complete and that $f\colon X \to X$ is* (ULC)*.*

(i) *If X is connected, then f has a unique fixed point.*
(ii) *If X has a finite number of components, then f has a periodic point, that is, $f^{(n)} = f \circ \cdots \circ f$ has a fixed point for some $n > 0$.*

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[1] So, a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ is (PC) if, and only if, $|f'(x)| < 1$ for every $x \in \mathbb{R}$, see Fact 3.1.

[2] The notion of (uPC) maps was introduced by Holmes [9], where it was called *local radial contraction*. (See also [10,12].) The term *radial* is often used elsewhere in mathematics and we find the adverb *pointwise* to be more adequate for this notion, see for example [8, p 104].

[3] In [2] the property (PC) is denoted (LRC), for *locally radially contractive*, and (uPC) is denoted as (uLRC).