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S_2 and the Fréchet property of free topological groups [☆]

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ABSTRACT

Let $F(X)$ denote the free topological group over a Tychonoff space X, $F_n(X)$ denote the subspace of $F(X)$ that consists of all words of reduced length $\leq n$ with respect to the free basis X for every non-negative integer n and $E_n(X) = F_n(X) \setminus F_{n-1}(X)$ for $n \geq 1$. In this paper, we study topological properties of free topological groups in terms of Arens' space S_2. The following results are obtained.
(1) If the free topological group $F(X)$ over a Tychonoff space X contains a non-trivial convergent sequence, then $F(X)$ contains a closed copy of S_2, equivalently, $F(X)$ contains a closed copy of S_ω, which extends [6, Theorem 1.6].
(2) Let X be a topological space and $A = \{n_1, ..., n_i, ...\}$ be an infinite subset of $\mathbb{N}$. If $C = \bigcup_{i \in \mathbb{N}} E_{n_i}(X)$ is κ-Fréchet–Urysohn and contains no copy of S_2, then X is discrete, which improves [15, Proposition 3.5].
(3) If X is a μ-space and $F_5(X)$ is Fréchet–Urysohn, then X is compact or discrete, which improves [15, Theorem 2.4].
At last, a question posed by K. Yamada is partially answered in a shorter alternative way by means of a Tanaka's theorem concerning Arens' space S_2.

1. Introduction

In 1941, the free topological group $F(X)$ over a Tychonoff space X in the sense of Markov was introduced [9]. Topologists discussed various topological properties on free topological groups, where sequentiality and the Fréchet property, as important topological properties, were investigated.

In 2014, F. Lin, C. Liu [6] showed the following.

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Theorem 1.1. *([6, Theorem 1.6]) If the free topological group $F(X)$ over a Tychonoff space X is a sequential space, then either X is discrete or $F(X)$ contains a copy of S_ω.*

In 2002, K. Yamada [15] investigated the Fréchet property of the subspace $F_n(X)$ of the free topological group $F(X)$ over a metrizable space X, where $F_n(X)$ denotes the subspace of $F(X)$ that consists of all words of reduced length $\leq n$ with respect to the free basis X for every non-negative integer n, and obtained the following results.

Theorem 1.2. *([15, Corollary 2.5]) Let X be a metrizable space. $F_3(X)$ is Fréchet–Urysohn if and only if the set of all non-isolated points of X is compact.*

Theorem 1.3. *([15, Theorem 2.4]) Let X be a metrizable space. If $F_5(X)$ is Fréchet–Urysohn, then X is compact or discrete.*

K. Yamada [15] posed following Question 1.4 and conjectured that the answer to this question is affirmative.

Question 1.4. Let X be a metrizable space. Is $F_4(X)$ is Fréchet–Urysohn if the set of all non-isolated points of X is compact?

F. Lin and C. Liu [6] tried to solve Question 1.4, however, there was a gap in the proof [6,7]. Hence Question 1.4 is still open.

In this paper, we shall make full use of the concept of Arens' space S_2 to establish our main results. This paper is organized as follows.

At first, we shall extend Theorem 1.1 by proving that if the free topological group $F(X)$ over a Tychonoff space X contains a non-trivial convergent sequence, then $F(X)$ contains a closed copy of S_2, equivalently, $F(X)$ contains a closed copy of S_ω.

Secondly, let X be a topological space and $A = \{n_1, ..., n_i, ...\}$ be an infinite subset of $\mathbb{N}$. If $C = \bigcup_{i\in\mathbb{N}} E_{n_i}(X)$ is κ-Fréchet–Urysohn and contains no copy of S_2, then X is discrete, which improves [15, Proposition 3.5].

Thirdly, we shall prove that if X is a μ-space and $F_5(X)$ is Fréchet–Urysohn, then X is compact or discrete, which improves Theorem 1.3.

Quite recently, K. Yamada [16], in a lengthy proof, proved that if X is a locally compact, metrizable space and the set of all non-isolated points of X is compact, then $F_4(X)$ is a k-space if and only if $F_4(X)$ is a Fréchet–Urysohn space. Further, if X is a locally compact, separable, metrizable space, then the set of all non-isolated points of X is compact if and only if $F_4(X)$ is a Fréchet–Urysohn space, which gave a partial answer to Question 1.4. In this paper, we shall present a shorter alternative way to prove the above result by means of a Tanaka's theorem concerning Arens' space S_2.

2. Preliminaries

A topological space X is called a *Fréchet* space or *Fréchet–Urysohn* (κ*-Fréchet–Urysohn*) space if for every $A \subset X$ (open subset $A \subset X$) and every $x \in \overline{A}$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of A converging to x. A topological space X is called a *sequential* space if a set $A \subset X$ is closed if and only if together with any sequence it contains its limits. Obviously, every Fréchet–Urysohn space is sequential (κ-Fréchet–Urysohn). A κ-Fréchet–Urysohn space need not be sequential [8].

Definition 2.1. ([1]) Let $X = \{0\} \cup \mathbb{N} \cup \mathbb{N}^2$. $\mathbb{N}^\mathbb{N}$ denotes the set of all functions from $\mathbb{N}$ to $\mathbb{N}$. For every $n, m, k \in \mathbb{N}$, put $V(n, m) = \{n\} \cup \{(n, k) : k \geq m\}$. For every $x \in \mathbb{N}^2$, let $\mathcal{B}(x) = \{\{x\}\}$. For every $n \in \mathbb{N}$, let $\mathcal{B}(n) = \{V(n, m) : m \in \mathbb{N}\}$. Let $\mathcal{B}(0) = \{\{0\} \cup \bigcup_{n\geq i} V(n, f(n)) : i \in \mathbb{N}, f \in \mathbb{N}^\mathbb{N}\}$. The topological space X, generated by the neighborhood system $\{\mathcal{B}(x)\}_{x\in X}$, is called *Arens'* space and denoted by S_2.