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Monotone maps on dendrites and their induced maps

Haithem Abouda, Issam Naghmouchi *

University of Carthage, Faculty of Sciences of Bizerte, Department of Mathematics, Jarzouna, 7021, Tunisia

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A continuum X is a dendrite if it is locally connected and contains no simple closed curve, a self mapping f of X is called monotone if the preimage of any connected subset of X is connected. If X is a dendrite and $f : X \to X$ is a monotone continuous map then we prove that any ω-limit set is approximated by periodic orbits and the family of all ω-limit sets is closed with respect to the Hausdorff metric. Second, we prove that the equality between the closure of the set of periodic points, the set of regularly recurrent points and the union of all ω-limit sets holds for the induced maps $\mathcal{F}_n(f) : \mathcal{F}_n(X) \to \mathcal{F}_n(X)$ and $\mathcal{T}_n(f) : \mathcal{T}_n(X) \to \mathcal{T}_n(X)$ where $\mathcal{F}_n(X)$ denotes the family of finite subsets of X with at most n points, $\mathcal{T}_n(X)$ denotes the family of subtrees of X with at most n endpoints and $\mathcal{F}_n(f) = 2^f_{|\mathcal{F}_n(X)}$, $\mathcal{T}_n(f) = 2^f_{|\mathcal{T}_n(X)}$, in particular there is no Li–Yorke pair for these maps. However, we will show that this rigidity in general is not exhibited by the induced map $\mathcal{C}(f) : \mathcal{C}(X) \to \mathcal{C}(X)$ where $\mathcal{C}(X)$ denotes the family of sub-continua of X and $\mathcal{C}(f) = 2^f_{|\mathcal{C}(X)}$, we will discuss an example of a homeomorphism g on a dendrite S which is dynamically simple whereas its induced map $\mathcal{C}(g)$ is ω-chaotic and has infinite topological entropy.

1. Introduction

Let $\mathbb{Z}_+$ and $\mathbb{N}$ be the sets of non-negative integers and positive integers respectively. Let X be a compact metric space with metric d and $f : X \longrightarrow X$ be a continuous map. Denote by f^n the n-th iterate of f; that is, $f^0 = \mathrm{id}_X$: the Identity and $f^n = f \circ f^{n-1}$ if $n \geq 1$. For any $x \in X$ the subset $O_f(x) = \{f^n(x) : \ n \in \mathbb{Z}_+\}$ is called the f-orbit of x. A point $x \in X$ is called *periodic* if $f^n(x) = x$ for some $n > 0$. When $n = 1$ we say that x is a *fixed point*. The period of p is the least natural number m such that $f^m(p) = p$. A subset A of X is called f*-invariant* (resp. strongly f-invariant) if $f(A) \subset A$ (resp. $f(A) = A$). It is called *a minimal set of* f if it is non-empty, closed, f-invariant and minimal (in the sense of inclusion) for these properties.

* Corresponding author.
E-mail addresses: haithem.abouda@yahoo.fr (H. Abouda), issam.naghmouchi@fsb.rnu.tn, issam.nagh@gmail.com (I. Naghmouchi).

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Note that a nonempty closed set $M \subseteq X$ is minimal if and only if the orbit of every point from M is dense in M. The ω-limit set of a point x is the set

$$\omega_f(x) = \{y \in X : \exists\ n_i \in \mathbb{N}, n_i \to \infty, \lim_{i\to\infty} d(f^{n_i}(x), y) = 0\}$$
$$= \bigcap_{n\in\mathbb{N}} \overline{\{f^k(x) : k \geq n\}}.$$

The set $\omega_f(x)$ is a non-empty, closed and strongly invariant set. If $\omega_f(x)$ is finite then it is a periodic orbit. If $\omega_{f^m}(x)$ is finite for some $m \in \mathbb{N}$ then $\omega_f(x)$ is also finite (see [8] for more details).

A point $x \in X$ is said to be:

- *recurrent* if for every open set U containing x there exists $n \in \mathbb{N}$ such that $f^n(x) \in U$ (equivalently, $x \in \omega_f(x)$).
- *regularly recurrent* if for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $d(x, f^{kN}(x)) < \varepsilon$ for all $k \in \mathbb{N}$.

It is known that if x is regularly recurrent then $\omega_f(x)$ is a minimal set (see [8, Proposition 5, Chapter V]).

Let $\mathrm{Fix}(f)$, $\mathrm{P}(f)$, $\mathrm{RR}(f)$, $\mathrm{R}(f)$ and $\Lambda(f)$ denote the set of fixed points, periodic points, regularly recurrent, recurrent points and the union of all ω-limit sets respectively. Then we have the following inclusion relation $\mathrm{Fix}(f) \subset \mathrm{P}(f) \subset \mathrm{RR}(f) \subset \mathrm{R}(f) \subset \Lambda(f)$.

The notion of *topological entropy* for a continuous self-map of a compact metric space was first introduced by Adler, Konheim and McAndrew in [4]. We recall here the equivalent definition formulated by Bowen [10], and independently by Dinaburg [11]: Let (X, d) be a compact metric space and let $f : X \to X$ be a continuous map. Fix $n \geq 1$ and $\varepsilon > 0$. A subset E of X is called (n, f, ε)-separated if for every two different points $x, y \in E$, there exists $0 \leq j < n$ with $d(f^j(x), f^j(y)) > \varepsilon$. Denote by $sep(n, f, \varepsilon)$ the maximal possible cardinality of an (n, f, ε)-separated set in X. Then the topological entropy of f is defined by

$$h(f) = \lim_{\varepsilon\to 0} \limsup_{n\to+\infty} \frac{1}{n}\ \ln\ sep(n, f, \varepsilon).$$

A pair $(x, y) \in X \times X$ is called *proximal* if $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$. If $\limsup_{n\to\infty} d(f^n(x), f^n(y)) = 0$, (x, y) is called *asymptotic*. A pair (x, y) is called a *Li–Yorke pair* (of f) if it is proximal but not asymptotic. A subset S of X containing at least two points is called a *Li–Yorke scrambled set* (of f) if for any $x, y \in S$ with $x \neq y$, (x, y) is a Li–Yorke pair. We say that f is *Li–Yorke chaotic* if there exists an uncountable scrambled set of f. It is proved in [7] that if $h(f) > 0$ then f is Li–Yorke chaotic.

A subset S of X containing at least two points is called an *ω-scrambled* set for f if for any $x, y \in S$ with $x \neq y$ the following conditions are fulfilled:

(i) $\omega_f(x) \setminus \omega_f(y)$ is uncountable,
(ii) $\omega_f(x) \cap \omega_f(y) \neq \emptyset$,
(iii) $\omega_f(x) \setminus \mathrm{P}(f) \neq \emptyset$.

The map f is called *ω-chaotic* if there is an uncountable ω-scrambled set [17]. Following Theorem 1.4 and Corollary 2.6 from [19] the definition of ω-scrambled set is reduced only to conditions (i) and (ii) when the space X is either a completely regular continuum (i.e. every non-degenerate sub-continuum of X has non-empty interior) or a zero-dimensional compact space.

It is noteworthy that ω-chaos is not related to the topological entropy. The paper [25, Example 4.3] provides an example of a map which is ω-chaotic with zero topological entropy. Surely, positive topological entropy is not enough to imply ω-chaos, since there is known example of minimal map with positive topological entropy [26].