Topology and its Applications 204 (2016) 149–156

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Lipschitz subspaces of $C(K)$

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A R T I C L E   I N F O

Article history:
Received 12 June 2015
Received in revised form 4 February 2016
Accepted 4 March 2016

MSC:
26A16
54C30
54C50
54E40

Keywords:
Lipschitz map
Metric space
Lipschitz manifold
Universal Banach space

A B S T R A C T

Let K be an uncountable metric compact space. It is well known that $C(K)$ is isometrically universal for the separable Banach spaces, but the continuous functions that compose the isometric image of finite dimensional spaces are typically far from being Lipschitz. We prove that the possibility of embedding Euclidean spaces $\mathbb{R}^n \hookrightarrow C(K)$ in such a way that the image in $C(K)$ is made of Lipschitz functions is tightly related to the dimension (topological or Hausdorff) of K.

1. Introduction

Throughout the paper all the Banach spaces considered are real. We shall denote by K a compact Hausdorff space, and $C(K)$ will be the Banach space of real continuous functions defined on K endowed with the supremum norm. The real unit interval is denoted by $\mathbb{I}$. We shall consider $\mathbb{I}$ and its finite powers with the Euclidean distance. As usual, if X is a Banach space we shall denote by B_X its closed unit ball, and by S_X its unit sphere. For any unexplained concepts or notations about Banach spaces we address the reader to [5] or [16].

A classical result of Banach and Mazur [5, Theorem 5.8] says that $C(\mathbb{I})$ is isometrically universal for the class of separable Banach spaces. In particular, the Euclidean spaces $(\mathbb{R}^n, \|\cdot\|_2)$ can be found isometrically as

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http://dx.doi.org/10.1016/j.topol.2016.03.003

subsets of functions defined on $\mathbb{I}$. For $n = 2$ an isometric embedding $J : \mathbb{R}^2 \to C(\mathbb{I})$ can be written explicitly as $J(x_1, x_2)(t) = x_1 \cos(\pi t) + x_2 \sin(\pi t)$, using C^∞ functions. As we see later, an isometric embedding of $\mathbb{R}^3$ cannot be written explicitly using such simple functions. In fact, Peano curves are needed as was first noticed in 1957 by Donoghue [4]. However, $\mathbb{R}^3$ is isometrically embedded into $C(\mathbb{I}^2)$ by means of the formula

$$J(x_1, x_2, x_3)(t, s) = x_1 \cos(\pi t) \cos(\pi s) + x_2 \sin(\pi t) \cos(\pi s) + x_3 \sin(\pi s).$$

We will see that the possibility of finding an "easy formula" for an isometric embedding of $\mathbb{R}^n$ into $C(K)$ is related to the *dimension* of K.

If K_1 and K_2 are uncountable metrizable compacta, then $C(K_1)$ and $C(K_2)$ are isomorphic by Milutin's theorem [16, III.D.19]. These Banach spaces cannot be isometric unless K_1 and K_2 are homeomorphic. On the other hand, $C(K_1)$ and $C(K_2)$ are universal spaces for the class of separable spaces in the isometric category. In particular $C(K_1)$ contains an isometric copy of $C(K_2)$ and vice versa. In particular, that means that it is not possible to distinguish between K_1 and K_2 by isometric embeddings of test spaces.

Our idea is to relate properties of a compact K to the existence of isometric embeddings $J : X \to C(K)$ of finite dimensional linear spaces X such that the set $J(X)$ is composed of "nice" functions. Here nice will mean Lipschitz at least, and the requirement of finite dimension is necessary. Indeed, it is easy to see that if the isometric embedding $J(X)$ is composed of Lipschitz functions, then X must be of finite dimension (Proposition 2.1). The next result shows the relation between K and the existence of nice embeddings of the Euclidean spaces.

Theorem 1.1. *Let (K, d) be an uncountable metric compact space and $n \in \mathbb{N}$. The following are equivalent:*

(i) *There is an onto Lipschitz mapping $\phi : K \to \mathbb{I}^n$.*
(ii) *$C(K)$ contains an isometric copy of any $(n+1)$-dimensional Banach space made of Lipschitz functions.*
(iii) *$C(K)$ contains an isometric copy of the Euclidean space $(\mathbb{R}^{n+1}, \|\cdot\|_2)$ made of Lipschitz functions.*

Moreover, if K is a Lipschitz manifold, then statements (i), (ii) and (iii) are also equivalent to

(iv) *The dimension of K is at least n.*

We follow [11] for the definition of Lipschitz manifold (with boundary). A separable metric space is called a *Lipschitz manifold* (of dimension n) if every point has a closed neighborhood which is Lipschitz homeomorphic to $\mathbb{I}^n$, that is, there is a Lipschitz bijective mapping whose inverse is Lipschitz too. We may apply our result as well to topological manifolds. Indeed, Sullivan [14] proved that n-dimensional topological manifolds have a Lipschitz structure for $n \neq 4$. Nevertheless, a fixed metric on K is needed since topologically equivalent metrics on K are in general not Lipschitz equivalent. If K is neither a topological or a Lipschitz manifold, we may still obtain information about K from the previous result using the Hausdorff dimension. Indeed, statement (i) clearly implies that the Hausdorff dimension of K is greater or equal than n (see [6, Corollary 2.4]). On the other hand, a recent result of Keleti, Máthé and Zindulka [8] says that if the Hausdorff dimension of K is strictly greater than n, then statement (i) holds. Unfortunately, the existence of a Lipschitz mapping onto a cube does not characterize the Hausdorff dimension as showed by the example constructed by Vitušhkin, Ivanov and Melnikov [15]. If K is ultrametric, then statement (i) implies that the Hausdorff dimension is at least n by another result of [8]. In the following, $\dim_H(K)$ will denote the Hausdorff dimension of K.

The smooth embedding of a smooth compact manifold into some $\mathbb{R}^N$ (see e.g. [10, Theorem 3.21]) induces on it a metric and a structure of a Lipschitz manifold. This structure is unique because two metrics obtained in the same way are Lipschitz equivalent (indeed, apply the compactness to the fact that both metric spaces