Topology and its Applications 204 (2016) 168–174

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Lagrange stability and asymptotic periods

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ARTICLE INFO

Article history:
Received 22 October 2015
Received in revised form 7 March 2016
Accepted 8 March 2016

MSC:
37B25
34D05
54D99

Keywords:
Lagrange stability
Asymptotic period
ω-Limit set
Proper metric space

ABSTRACT

In this article we describe selected topological and dynamical properties of asymptotically periodic motions in continuous dynamical systems (flows). The main result is to show Lagrange stability (i.e. the closure of positive orbit is compact) of such motion with the aid of topological properties of limit sets. Two sufficient conditions for this kind of stability are provided: the value of asymptotic period (our approach to this notion differs from the usual one) and proper metric space property.

1. Topological background

Compactness and connectedness are among fundamental properties of limit sets that we expect in the theory of continuous dynamical systems. They can be derived using various properties of orbits, neighbourhoods of a limit set or from assumptions imposed on a phase space of a dynamical system. The most elementary case is when X is (locally) compact.

The importance of studying topological properties of limit sets is significant; one can describe dynamical properties of limit sets and their neighbourhood based on such attributes. Various connections are featured in [2]. Bacciotti and Kalouptsidis [1] provided results connecting the topology and dynamical properties, such as attraction and stability in dynamical polysystems.

There are several examples replacing compactness with weaker assumptions. One such approach is present in [3], where both limit sets and prolongation limit sets are investigated in the case of c-first countable spaces. A more specific approach is given in [1]. In this paper we present different criteria that describe topological properties of a metric space and dynamical properties of an orbit [4].

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http://dx.doi.org/10.1016/j.topol.2016.03.005

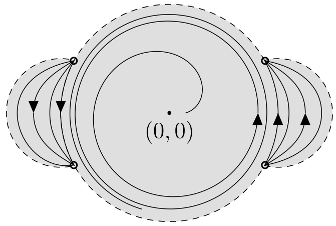

Fig. 1. Sketch of a system defined in Example 2.1.

The paper is organised as follows. In Section 2 we introduce definitions and notations and discuss topological properties of limit sets from the point of view of assumptions set on a phase space and on the dynamical system. This approach is important for the stability of a motion. In Section 3 we establish the main results implying topological properties as well as Lagrange stability and attraction. We also provide counterexamples for converse statements and present simple applications of results. We conclude the paper with few remarks, provided in Section 4.

2. Preliminaries and the example

The reader is assumed to know elementary definitions concerning continuous dynamical systems (flows) and the topology of metric spaces. However, all important definitions used in this article are introduced.

Throughout this article ϕ denotes a (local) dynamical system on a metric space (X, d), $\omega(x)$ is the omega-limit set of $x \in X$ and $o^+(x)$ is the positive orbit of x. Whenever we introduce the polar coordinate system, r is a radius and ϑ is an angle. We denote $I_x := \{t \in \mathbb{R} \mid (t, x) \in \operatorname{dom}\phi\}$ (if ϕ is a local flow, then I_x is an open interval). We write $B(x, r)$ for an open ball centred at x with radius r; the corresponding closed ball is denoted by $\overline{B}(x, r)$. Recall that the point x (the motion along x and consequently, its orbit) is *Lagrange stable* if the set $\overline{o^+(x)}$ is compact.

Our points of reference are basic properties of ω-limit sets. It is known that in arbitrary metric space, the set $\omega(x)$ and every connected component of this set is closed and invariant. In case of compact metric spaces, $\omega(x)$ is non-empty, compact and connected. Finally, in locally compact spaces $\omega(x)$ is connected whenever it is compact [2]. A natural question arising from these properties is whether compactness assumptions of X can be replaced with different, weaker conditions to the space. Consider the following example.

Example 2.1. Set $X := B((0,0),1) \cup B((1,0),1/2) \cup B((-1,0),1/2)$ and equip it with Euclid metric. Consider a local dynamical system ϕ defined on X. We introduce polar coordinates for the case $\|u\| \le 1$. We set the motion using the following system of equations:

$$\begin{cases} r' = r(1-r), \\ \vartheta' = 1. \end{cases}$$

If $\|u\| = 1$, then orbits of points $(1, 0)$ and $(-1, 0)$ are defined on a bounded and open interval $I_{(-1,0)} = I_{(1,0)} = (-\arccos 7/8, \arccos 7/8)$.

Consider the family of circles $\{C_w \mid w \in (0,1)\}$, each C_w is given by the equation $x^2 - 2xw + y^2 = 1 - 7w/4$. Pick any point $u \in X$ such that $\|u\| > 1$ and the abscissa of u is positive. The orbit of this point coincides with $C_w \cap B((1,0),1/2)$, where the set C_w is chosen in such a way that $u \in C_w$ for some unique $w \in (0,1)$. We assume that the motion on each orbit progresses with a constant angular velocity (see Fig. 1). In this case I_u are open and bounded intervals (we can assume that $I_u = (-\arccos 7/8, \arccos 7/8)$ for all u with the ordinate equal to zero, in remaining cases intervals I_u have the length equal to $2\arccos 7/8$ and their ending points are shifted). Using analogous argument we define ϕ inside $B((-1,0),1/2) \setminus B((0,0),1)$.

It follows from the construction that for $u = (1/2, 1/2)$ the set $\omega(u)$ is neither compact nor connected.