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A criterion for the Legendrian simplicity of the connected sum ☆

Byung Hee An

Center for Geometry and Physics, Institue for Basic Science (IBS), Pohang 37673, Republic of Korea

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ABSTRACT

In this paper, we provide the necessary and sufficient conditions for the connected sum of knots in S^3 to be Legendrian simple.

1. Introduction

Throughout this paper, we consider oriented Legendrian knots in S^3 with the standard tight contact structure ξ_{std}.

Legendrian knot theory is the study of Legendrian knots up to Legendrian isotopy which is much more restricted compared to the smooth or piecewise-linear isotopy in the classical knot theory. Due to this restriction, for given topological knot type K, there are infinitely many inequivalent Lengendrian isotopy classes of topological knot type K, and we denote by $\mathcal{L}(K)$ the set of Legendrian isotopy classes of K. There are two well-known Legendrian knot invariants, *Thurston–Bennequin number* $tb(L)$ and *rotation number* $r(L)$ that can be used to classify Legendrian knots in $\mathcal{L}(K)$, and they are called *classical invariants*. A topological knot type K is said to be *Legendrian simple* if Legendrian knots in $\mathcal{L}(K)$ are classified by the classical invariants. Refer to [8] for details.

The Legendrian simplicity for unknot has been shown by Eliashberg and Fraser in 1995 [3], and for figure-eight and torus knots by Etnyre and Honda in 2001 [4]. In their follow-up paper [5], Etnyre and

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E-mail address: anbyhee@ibs.re.kr.

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Honda provided a complete combinatorial description for the connected sum of Legendrian knots, and proved that the Legendrian simplicity is not closed under the connected sum.

Etnyre and Honda also proved in [6] that the cabling operation preserves the Legendrian simplicity under the *uniform thickness property* (UTP). The important benifits of UTP are that it will be preserved not only by cabling with the certain condition, but also by the connected sum. Hence, if started with Legendrian simple and UTP knot, one may produce the arbitrarily many Legendrian simple knots by the iterated cabling as studied in [7,9,10]. However, as shown by Etnyre and Honda, Legendrian simplicity may or may not be preserved under connected sum, and therefore attempts to cable after connected sum may fail.

This paper provides the necessary and sufficient conditions for the connected sum of knot types to be Legendrian simple as follows.

Theorem 1. *Let $K = (\#^{a_1} K_1)\# \dots (\#^{a_n} K_n)$ be a connected sum decomposition of K with pairwise distinct prime knots. Then K is Legendrian simple if and only if all K_i's are Legendrian simple and one of the following is satisfied:*

(1) $|Peak(K_i)| = 1$ *for all* i*;*
(2) *There exists only one* i *such that* $|Peak(K_i)| = 2$*,* $a_i \geq 2$ *and* $|Peak(K_j)| = 1$ *if* $j \neq i$*;*
(3) *There exists only one* i *such that* $|Peak(K_i)| \geq 3$*,* $a_i = 1$ *and* $|Peak(K_j)| = 1$ *if* $j \neq i$*.*

Here, $Peak(K_i)$ *is the set of Legendrian knots in* $\mathcal{L}(K_i)$ *which can not be destabilized in either ways.*

Therefore this theorem provides the way to produce the new Legendrian simple knots. Moreover, this can be used with the the cabling construction, as well.

The rest of this paper consists of the following. In section 2, we introduce the basic notions and briefly review the known results about Legendrian connected sum. In section 3, we prove the main result.

2. Preliminaries

2.1. Basic notions

A *topological oriented knot*, simply a *knot* K from now on, is a smooth embedding of S^1 into S^3, and a *knot type* $[K]$ is a smooth isotopy class of K. The natural orientation of K comes from $d\theta$ where S^1 is parametrized by θ. A knot K is *trivial* or an *unknot* if K bounds an embedded disc in S^3.

A *Legendrian knot* L is an embedding of S^1 into (S^3, ξ_{std}) such that L is everywhere tangent to the contact plane ξ_{std}. Here ξ_{std} is the standard contact structure on S^3. Similarly, a *Legendrian knot type* $[L]$ is the Legendrian isotopy class of L. From now on, we simply use K, L instead of $[K]$, $[L]$ to denote knot types unless any ambiguity occurs.

For given knot type K, we denote by $\mathcal{L}(K)$ the set of Legendrian knot types having topological knot type K. Then as mentioned before, any two elements in $\mathcal{L}(K)$ can be connected with a sequence of two special types of isotopies, called *positive and negative stabilizations* $S_\pm$, and their inverses. These two stabilizations are commutative in the sense that $S_+(S_-(L)) = S_-(S_+(L))$ for any Legendrian knot L. Diagrammatically, $S_\pm$ is as depicted in Fig. 1. Hence we can regard $\mathcal{L}(K)$ as a connected, directed graph by adding directed edges $(L, S_+(L))$ and $(L, S_-(L))$ for each $L \in \mathcal{L}(K)$.

The classical invariants, Thurston–Bennequin number $tb(L)$ and the rotation number $r(L)$, for a Legendrian knot L change as follows under stabilizations.

$$tb(S_\pm(L)) = tb(L) - 1, \quad r(S_\pm(L)) = r(L) \pm 1.$$