Topology and its Applications 204 (2016) 185–197

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Discrete Euler integration over functions on finite categories ☆

Kohei Tanaka

Institute of Social Sciences, School of Humanities and Social Sciences, Academic Assembly, Shinshu University, Japan

ARTICLE INFO

Article history:
Received 11 September 2015
Accepted 13 March 2016

MSC:
55P10
46M20

Keywords:
Euler integration
Euler characteristic
Finite categories
Sensor networks

ABSTRACT

This paper provides the theory of integration with respect to Euler characteristics of finite categories. As an application, we use sensors to enumerate the targets lying on a poset. This is a discrete analogue to Baryshnikov and Ghrist's work on integral theory using topological Euler characteristics.

1. Introduction

It has long been known that the Euler characteristic is an important homotopy invariant of a space. By regarding it as a topological measure, we can derive the theory of the integral with respect to the Euler characteristic (*Euler integration*). The notion of Euler integration or Euler calculus was introduced independently by Viro [12] and Schapira [6] in the late 1980's. At the end of 2000's, Baryshnikov and Ghrist applied it to sensor networks [2]. They established a way to use sensors to enumerate targets in a filed. Let us review briefly a simple case that they considered.

Consider a situation in which there are a finite number of targets T lying on a topological space X. Assume that each point of X has a sensor recording the nearby targets, and each target $t \in T$ has a contractible target support:

$$U_t = \{x \in X \mid \text{the sensor at } x \text{ detects } t\}.$$

☆ This work was supported by JSPS KAKENHI Grant Number 15K17535, and CREST, JST Mathematics (15656429). The author would like to thank Yasuhide Numata for his helpful comments and suggestions.

E-mail address: tanaka@shinshu-u.ac.jp.

http://dx.doi.org/10.1016/j.topol.2016.03.012
0166-8641/

The sensors return the counting function $h : X \to \mathbb{N} \cup \{0\}$ given by the number of detectable sensors at each point:

$$h(x) = \{t \in T \mid x \in U_t\}^\sharp.$$

Then, we can enumerate the targets by integrating with respect to the Euler characteristic χ (Theorem 3.2 of [2]):

$$T^\sharp = \int_X h d\chi.$$

Our goal in this paper is to show a discrete analogue of the above. The Euler characteristic is defined not only for topological spaces, but also for finite combinatorial objects, such as posets [5], groupoids [1], and categories [4]. We will focus on the Euler characteristics of finite categories, which is the most general case (note that a poset can be thought of as an acyclic category with at most one morphism between any pair of objects). By using this instead of topological Euler characteristic, we can derive a discrete version of Euler integration. We treat certain rational-valued functions on objects of a category as integrable functions, since Euler characteristics of finite categories take rational values.

As an application of discrete Euler integration, we consider the counting problem in a network flowing in only one direction (such as transmission of electricity, streams of water or rivers, and acyclic traffic). We propose a way to use sensors to enumerate the targets lying on a one-way network.

The remainder of this paper is organized as follows. Section 2 prepares some necessary definitions and notations for Euler characteristics of finite categories; we use the system created by Leinster [4]. In Section 3, we introduce the class of integrable (*constructible*) functions on a finite category. Following this, we define the Euler integration of definable functions, and investigate its properties. Section 4 presents an application with sensor networks.

2. Euler characteristics of finite categories

The Euler characteristic of a finite category was introduced by Leinster [4]; it is a generalization of the concept of Möbius inversion [5] for posets.

Definition 2.1. Suppose that C is a finite category consisting of a finite number of objects and morphisms. We denote the set of objects of C by $\mathrm{ob}(C)$, and the set of morphisms from x to y by $C(x, y)$.

(1) The *similarity matrix* of C is the function $\zeta : \mathrm{ob}(C) \times \mathrm{ob}(C) \to \mathbb{Q}$, given by the cardinality of each set of morphisms: $\zeta(a, b) = C(a, b)^\sharp$.
(2) Let $u : \mathrm{ob}(C) \to \mathbb{Q}$ denote the column vector with $u(a) = 1$, for any object a of C. A *weighting* on C is a column vector $w : \mathrm{ob}(C) \to \mathbb{Q}$ such that $\zeta w = u$, and dually, a *coweighting* on C is a row vector $v : \mathrm{ob}(C) \to \mathbb{Q}$ such that $v\zeta = u^*$, where u^* is the transposition of the matrix u.

Note that we have

$$\sum_{i \in \mathrm{ob}(C)} w(i) = u^* w = v\zeta w = vu = \sum_{j \in \mathrm{ob}(C)} v(j),$$

if both a weighting and a coweighting exist. Moreover,

$$\sum_{i \in \mathrm{ob}(C)} w(i) = u^* w = v\zeta w = v\zeta w' = u^* w' = \sum_{i \in \mathrm{ob}(C)} w'(i),$$

for two (co)weightings w and w' on C. This guarantees the following definition of the Euler characteristic.