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Combinatorial embedding of chain transitive zero-dimensional systems into chaos

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ABSTRACT

In this paper, we construct examples of transitive Li–Yorke chaotic zero-dimensional topological dynamical systems, which should be able to accommodate of a variety of subsystems. In fact, we show that a zero-dimensional chain transitive dynamical system can be embedded into a transitive Li–Yorke chaotic system with a dense scrambled set, and with the same topological entropy. Our work reflects the notion of uniform chaos, which was recently developed. In addition, our work reflects the recent work of invariance of scrambled sets in general topological setting. In these works, Mycielski sets play an essential role. Actually, we concretely construct a Mycielski set K such that it is densely uniformly chaotic, and also invariant in both directions. Furthermore, every point in K is positively and negatively transitive. The uniform proximality and recurrence of K are also bidirectional.

1. Introduction

A pair (X, f) of a compact metric space X and a continuous surjection $f : X \to X$ is called a *topological dynamical system*. A topological dynamical system (X, f) is *topologically transitive* or just *transitive*, if for any non-empty open sets $U, V \subseteq X$, there exists an integer $n > 0$ such that $f^n(U) \cap V \neq \emptyset$. It is a fact that when (X, f) is transitive, then there exists a point $x \in X$ such that the positive orbit $\{ f^i(x) \mid i \geq 0 \}$ is dense in X. A point with this property is called a *transitive point*. A topological dynamical system (X, f) is called a *zero-dimensional system*, if X is totally disconnected. In [5], we presented a way to express every zero-dimensional system combinatorially by a sequence of graph coverings (see §2.2). A topological dynamical system (X, f) is said to be *chain transitive* if for any $\epsilon > 0$ and any pair $(x, y) \in X \times X$, there exists a finite sequence $(x_0 = x, x_1, x_2, \dots, x_l = y)$ such that $d(f(x_i), x_{i+1}) < \epsilon$ for all $0 \leq i < l$. A pair $(x, y) \in X^2 \setminus \Delta_X$ is said to be proximal, if

$$\liminf_{n \to +\infty} d(f^n(x), f^n(y)) = 0.$$

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A pair (x, y) is said to be a Li–Yorke pair if (x, y) is proximal and (x, y) satisfies

$$\limsup_{n\to+\infty} d(f^n(x), f^n(y)) > 0.$$

A subset $K \subset X$ is said to be *scrambled* if any pair $(x, y) \in K^2 \setminus \Delta_K$ is a Li–Yorke pair. If there exists an uncountable scrambled set, then the system is said to be a Li–Yorke chaotic system. Note that if a proximal pair (x, y) with $x \neq y$ is recurrent in the system $(X \times X, f \times f)$, then it is a Li–Yorke pair. Akin et al. [1] presented a stronger notion of chaos called the uniform chaos, and also presented a criterion for chaos (Theorem 3.1 of [1]). Actually, they posed the notion of a uniformly chaotic set (see Definition 2.1). They constructed a (dense) uniformly chaotic set as a Mycielski set for many cases of Li–Yorke chaotic systems. Later, in [4], Oprocha and Zhang have shown that systems that have weakly mixing sets are uniformly chaotic. A nonempty subset K of a compact space X is a Mycielski set if it can be represented as a countable union of Cantor sets. A system that contains a uniformly chaotic set is said to be uniformly chaotic, and a system that contains a dense uniformly chaotic set is said to be densely uniformly chaotic.

On the other hand, Yuan and Lü in [8], and Tan in [7], without the assumption of compactness, investigated Mycielski invariant scrambled sets. Let (Y, g) be an arbitrary chain transitive zero-dimensional system. In this paper, we construct a chaotic zero-dimensional system (X, f) that contains (Y, g), has the same topological entropy, and has a dense scrambled set. Precisely, we show the following:

Theorem A. *Let (Y, g) be a chain transitive zero-dimensional system. Then, there exists a densely uniformly chaotic zero-dimensional system (X, f) with a dense uniformly chaotic invariant set $K \subset X \setminus Y$ such that*

(a) *the restriction $f|_{X\setminus Y}$ is a homeomorphism,*
(b) *$K = \bigcup_{N\geq 1} C_N$, where $C_1 \subset C_2 \subset \cdots$ is an increasing sequence of Cantor sets,*
(c) *each C_N is both positively and also negatively uniformly proximal,*
(d) *each C_N is both positively and also negatively uniformly recurrent,*
(e) *$f(K) = K$,*
(f) *K is dense in X,*
(g) *each $x \in K$ is positively and also negatively transitive,*
(h) *there exists a fixed point p and an $x_0 \neq p$ such that $\lim_{n\to+\infty} f^n(x_0) \to p$.*

The calculation of the topological entropy of (X, f) is done in Theorem 3.11. By (b), (c), (d), (e), and (f), K is a dense uniformly chaotic invariant set. The last condition implies that (X, f) is not uniformly rigid. Recently, in Theorem C of [3], Foryś et al. have shown that, if a non-trivial dynamical system (X, f) is transitive, then (X, f) contains a dense Mycielski invariant δ-scrambled set for some $\delta > 0$ if and only if it has a fixed point and is not uniformly rigid. By this result, the (X, f) obtained in our main theorem should have an invariant δ-scrambled set for some $\delta > 0$.

2. Preliminaries

Let $\mathbb{Z}$ be the set of all integers, and $\mathbb{N}$ be the set of all non-negative integers. For integers $a < b$, the intervals are denoted by $[a, b] := \{ a, a + 1, \ldots, b \}$.

2.1. Uniformly chaotic set

In this subsection, we introduce the notion presented by Akin et al. in [1]. Let (X, f) be a topological dynamical system. A subset $A \subset X$ is *uniformly recurrent* if for every $\epsilon > 0$ there exists an arbitrar-