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A correction to the paper "Injective mappings and solvable vector fields of Euclidean spaces"

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ABSTRACT

We construct a counterexample that disproves the claim of Theorem 0.2 of the paper "Injective mappings and solvable vector fields of Euclidean spaces", which appeared in Topology Appl. **136** (2004), 261–274.

1. Introduction

Given a C^∞ local diffeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$, it has been an interest of research along the years to find additional condition ensuring that F is globally injective. In $\mathbb{R}^2$, the non-injective local diffeomorphism $(x, y) \mapsto (e^x \cos y, e^x \sin y)$ shows that, in fact, more hypotheses are needed. It is not sufficient, for instance, to assume that F is a polynomial map, since in [8] it was constructed a non-injective polynomial map in $\mathbb{R}^2$ with non-zero Jacobian determinant, disproving the so called *real Jacobian conjecture*. Yet in the polynomial case, if the Jacobian determinant is a non-zero *constant*, then the injectivity of F is the famous *Jacobian conjecture* in $\mathbb{R}^n$, which is up to now an open problem. We address the reader to [9] for the very general result that F is a global diffeomorphism if and only if it is proper. We also mention a spectral condition in $\mathbb{R}^2$ given in [5] and, in a more general frame, in [4]. For the polynomial case and the Jacobian conjecture, we indicate [7].

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In [10] and [11] an approach using vector fields was given. To explain it, we introduce some definitions and notations.

We say that a vector field $\mathcal{X} : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is *globally solvable* if it is a surjective operator.

Letting $F = (F_1, \ldots, F_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a C^∞ map, we define n vector fields $\mathcal{V}_i : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$, $i = 1, \ldots, n$, as follows:

$$\mathcal{V}_i(\varphi) = \det D\left(F_1, \ldots, F_{i-1}, \varphi, F_{i+1}, \ldots, F_n\right),$$

for each $\varphi \in C^\infty(\mathbb{R}^n)$.

In $\mathbb{R}^2$, the following is true: if $F = (F_1, F_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is a local diffeomorphism and $\mathcal{V}_1$ or $\mathcal{V}_2$ is globally solvable, then F is injective. Indeed, if for instance $\mathcal{V}_1$ is globally solvable, then all the level sets of F_2 are connected, see [1] or [2] for proofs of this result. Then the injectivity of F follows, see Proposition 9 bellow.

We could think that a similar result remains true in higher dimensions. In this paper we give a counterexample for a generalization of this result, disproving the extension given in [10], and consequently in [11]. Precisely, we prove the following.

Theorem 1. *There exists a C^∞ non-injective local diffeomorphism $F : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathcal{V}_1$ and $\mathcal{V}_2$ are globally solvable.*

Theorem 0.2 of [10] claims that if a C^∞ local diffeomorphism $F : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ is such that Ω is *F-convex*, then F is injective. From the Definition 0.1 of [10], if $\mathbb{R}^3$ is *$\mathcal{V}_i$-convex for supports*, for $i = 1, 2$, then $\mathbb{R}^3$ is F-convex. According to Definition 0.0 of [10] and comments after it, $\mathbb{R}^3$ is $\mathcal{V}_i$-convex for supports if and only if $\mathcal{V}_i$ is globally solvable. Therefore our Theorem 1 disproves Theorem 0.2 of [10].

Moreover, we mention here the approach given in [12] and [13] (for dimension 2) and in [6] (for dimension n) in the case of F being a *polynomial* map from $\mathbb{C}^n$ to $\mathbb{C}^n$ and $\mathcal{V}_i$, $i = 1, \ldots, n$, being as above (but now complex vector fields) from E_n to E_n, where E_n is the space of the entire functions. It was proved that if $\mathcal{V}_i$ is surjective for $i = 1, \ldots, n-1$, then F is a polynomial automorphism. Our example given in Theorem 1 also shows that we can not expect a similar result for more general maps defined in $\mathbb{R}^n$.

The main tool to prove Theorem 1 is a relation between global solvability of the vector field $\mathcal{V}_i$ and connectedness of some suitable intersections of level sets of F_j, $j \neq i$. It is our Theorem 2 below. Actually we will use the sufficient condition given in Theorem 2 to prove the global solvability of $\mathcal{V}_1$ and $\mathcal{V}_2$ for the map F of Theorem 1.

Before stating the theorem, we introduce the notation $\mathscr{F}(f)$ to indicate the foliation of codimension 1 given by the connected components of the non-empty level sets of a C^∞ submersion $f : \mathbb{R}^3 \to \mathbb{R}$.

Theorem 2. *Let $F = (F_1, F_2, F_3) : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^∞ map satisfying $\det DF(x) \neq 0$ $\forall x \in \mathbb{R}^3$, and $i \in \{1, 2, 3\}$. If for each $j, k \in \{1, 2, 3\} \setminus \{i\}$, $j \neq k$, $L_j \in \mathscr{F}(F_j)$ and $c_k \in \mathbb{R}$, the set*

$$L_j \cap F_k^{-1}\{c_k\}$$

is connected, then $\mathcal{V}_i$ is globally solvable.

A version for $\mathbb{R}^n$ of Theorem 2 will soon appear in [2]. We remark that the converse of Theorem 2 is also true, see [2].

The paper is organized as follows. We begin section 2 explaining the general ideas for the construction of the counterexample given in Theorem 1. Then we give all the details of our construction culminating with the proof of Theorem 1, assuming Theorem 2. Finally, in section 3, we give some properties of the vector fields $\mathcal{V}_i$ and prove Theorem 2.