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Generalization of the Bolzano theorem for simplices

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A B S T R A C T

For an n-dimensional simplex in $\mathbb{R}^n$, a generalization of the Bolzano theorem is given. A proof based on the classical Knaster–Kuratowski–Mazurkiewicz covering lemma is obtained.

1. Introduction

The very important and pioneering Bolzano theorem (also called intermediate value theorem) states that [2,11]:

Bolzano's theorem: *If $f\colon [a, b] \subset \mathbb{R} \to \mathbb{R}$ is a continuous function and if it holds that $f(a)f(b) < 0$, then there is at least one $x \in (a, b)$ such that $f(x) = 0$.*

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Its first proofs, given independently by Bolzano in 1817 [2] and Cauchy in 1821 [5], were crucial in the procedure of arithmetization of analysis (which was a research program in the foundations of mathematics during the second half of the 19th century).

A straightforward generalization of Bolzano's theorem to continuous mappings of an n-cube (parallelotope) into $\mathbb{R}^n$ was proposed without proof by Poincaré in 1883 and 1884 in his work on the three-body problem [23,24]. The Poincaré theorem was soon forgotten and it has come to be known as "Miranda's theorem" [20] which partly explains the nomenclature "Poincaré–Miranda theorem" [18] as well as "Bolzano–Poincaré–Miranda theorem" [33].

The Bolzano–Poincaré–Miranda theorem states that [20,32,34]:

Bolzano–Poincaré–Miranda theorem: *Suppose that* $P = \{x \in \mathbb{R}^n \mid |x_i| < L, \text{ for } 1 \leqslant i \leqslant n\}$ *and let the mapping* $F_n = (f_1, f_2, \dots, f_n)\colon P \to \mathbb{R}^n$ *be continuous on the closure of* P *such that* $F_n(x) \neq \theta^n = (0, 0, \dots, 0)$ *for* x *on the boundary of* P*, and*

(*a*) $f_i(x_1, x_2, \dots, x_{i-1}, -L, x_{i+1}, \dots, x_n) \geqslant 0, \ \text{for} \ 1 \leqslant i \leqslant n,$

(*b*) $f_i(x_1, x_2, \dots, x_{i-1}, +L, x_{i+1}, \dots, x_n) \leqslant 0, \ \text{for} \ 1 \leqslant i \leqslant n.$

Then, there is at least one $x \in P$ *such that* $F_n(x) = \theta^n$.

Miranda in 1940 [20] showed that this theorem is equivalent to the Brouwer fixed point theorem [4]. It is worth noting that the Bolzano–Poincaré–Miranda theorem is closely related to important theorems in analysis and topology as well as it is an invaluable tool for verified solutions of numerical problems by means of interval arithmetic [14,21,22,27]. For a short proof as well as for a generalization of the Bolzano–Poincaré–Miranda theorem using topological degree theory see [34]. Also, for a generalization of this theorem to an infinite-dimensional setting see [26,27]. Furthermore, for interesting relations between the theorems of Bolzano–Poincaré–Miranda, Borsuk [3], Kantorovich[1] [13] and Smale[2] [30] with respect to the existence of a solution of a system of nonlinear equations, we refer the interested reader to [1].

In the paper at hand, a generalization of the Bolzano theorem for simplices is proposed. The obtained proof is based on the Knaster–Kuratowski–Mazurkiewicz lemma (KKM lemma for short, often called the KKM covering theorem or KKM covering principle). The KKM covering principle simply states that [15]:

KKM covering principle: *A family of* $(n + 1)$ *closed subsets covering an* n*-dimensional simplex and satisfying the Knaster–Kuratowski–Mazurkiewicz boundary conditions have a non-empty intersection.*

For a mathematical formulation of this statement the reader is referred to Lemma 2.1 below. The KKM lemma constitutes the basis for the proof of many theorems (including the famous Brouwer fixed point theorem). It is worth noting that three pioneering classical results, namely, the Brouwer fixed point theorem [4], the Sperner lemma [31], and the KKM lemma [15] are mutually equivalent in the sense that each one can be deduced from another.

The KKM lemma has numerous applications in various fields of pure and applied mathematics. In particular, among others, in the field of mathematical economics, the very important and pioneering extension of the KKM lemma due to Shapley[3] [28], customarily called the Knaster–Kuratowski–Mazurkiewicz–Shapley theorem (KKMS theorem for short), constitutes the basis for the proof of many theorems on the existence of solutions in game theory and in the general equilibrium theory of economic analysis. The Shapley's KKMS covering principle simply states that [25,28]:

[1] Nobel Laureate in Economic Sciences in 1975.

[2] Fields Medalist in 1966.

[3] Nobel Laureate in Economic Sciences in 2012.