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$2^{\aleph_0}$ ways of approaching a continuum with $[1, \infty)$

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ABSTRACT

In 2014, V. Martínez-de-la-Vega and P. Minc proved that, for an arbitrary nondegenerate metric continuum X, there is an uncountable collection $\mathcal{K}$ of topologically distinct metric compactifications of $[1, \infty)$, having X as the remainder. It is not clear without the continuum hypothesis that cardinality of $\mathcal{K}$ is $2^{\aleph_0}$. However, the continuum hypothesis is rarely necessary in the theory of metric continua. To support this assertion, presented here is an explicit construction of a compact metric space K with $2^{\aleph_0}$ mutually not homeomorphic components each of which is a compactification of $[1, \infty)$, having a copy of X as the remainder.
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1. Introduction

All spaces considered in this paper are metric. A *compactum* is a compact metric space, and a *continuum* is a connected compactum. Two continua are *incomparable* if none of them can be continuously mapped onto the other; otherwise the continua are *comparable*. If K is a compactum, then by 2^K we understand the space of all nonempty closed subsets of K with the Hausdorff metric, see [15, p. 53].

We say that Y is a *compactification* of the half-line $[1, \infty)$ with X as the *remainder* provided that Y and X are continua, $X \subset Y$, $R = Y \setminus X$ is homeomorphic to $[1, \infty)$, and R is dense in Y. In this context, we call R a *ray*. Metric compactifications of the real half-line have been extensively studied in continuum theory since the first half of the twentieth century. In particular, Waraszkiewicz [18] (1932) gave an example of $2^{\aleph_0}$ mutually incomparable compactifications of $[1, \infty)$ with a simple closed curve S^1 as the remainder; for a simpler proof see also [16]. Constructing large collections of mutually not homeomorphic or incomparable compactifications of the real half-line, with a given continuum X as the remainder, shows that X can be

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approached by a ray in many substantially different ways. In 1979, Russo [17] gave an example with similar properties to that of by Waraszkiewicz, except that S^1 was replaced by a simple triod. In 1993, Awartani constructed a similar example with S^1 was replaced by an arc [3], see also [1] and [2]. In each the above three examples the remainder is a Peano (i.e. locally connected) continuum. Recently, Bartoš, Marciňa, Pyrih and Vejnar [4] proved a significantly stronger result with S^1 replaced by an arbitrary Peano continuum.

In 2004, Martínez-de-la-Vega constructed a collection of $2^{\aleph_0}$ topologically distinct compactifications of $[1, \infty)$ with the pseudo-arc as the remainder [10], see also [9]. Notice that the compactifications constructed by Martínez-de-la-Vega are mutually not homeomorphic, but they are comparable. All incomparable continua are not homeomorphic, but the opposite is not true. Since incomparable continua differ from each other more profoundly than those that are just not homeomorphic, it is usually more desirable to construct incomparable examples, see [6,8,11], and [13] for more collections of incomparable continua. However, any two compactifications of the real half-line with the pseudo-arc as the remainder are comparable. Moreover, Illanes, Minc and Sturm proved recently that for any compactifications Y_1 and Y_2 of $[1, \infty)$ with the pseudo-arc as the remainder, each continuous surjection of the remainder of Y_1 onto the remainder of Y_2 can be extended to a continuous surjection of Y_1 onto Y_2, see [7, Corollary 1.5].

In 2014, Martínez-de-la-Vega and Minc [12] proved that, for an arbitrary nondegenerate metric continuum X, there is an uncountable collection $\mathcal{K}$ of topologically distinct metric compactifications of $[1, \infty)$, having X as the remainder. It is not clear without the continuum hypothesis that cardinality of $\mathcal{K}$ is $2^{\aleph_0}$. However, the continuum hypothesis is rarely necessary in the theory of metric continua. To support this assertion, we expand on methods used in [12] and [14] and give an explicit construction of a compact metric space K with $2^{\aleph_0}$ mutually not homeomorphic components each of which is a compactification of $[1, \infty)$, having a copy of X as the remainder. More precisely we prove the following theorem:

Theorem 1. *For each nondegenerate metric continuum X there is a compactum K such that*

(1) *components of K are mutually not homeomorphic and each of them is a compactification of $[1, \infty)$ with a copy of X as the remainder,*
(2) *the decomposition of K into its components is continuous and the decomposition space is homeomorphic to the Cantor set.*

The following corollary is a simple consequence of the above theorem.

Corollary 1. *For each nondegenerate metric continuum X there is a collection of $2^{\aleph_0}$ mutually not homeomorphic metric compactifications of $[1, \infty)$ with X as the remainder.*

By [7, Corollary 1.5], we cannot require in Theorem 1 and in the corollary that the elements of the collection be incomparable. Neither can we increase the cardinality of the collection, since there are only $2^{\aleph_0}$ topologically different metric continua. So, if the variety of ways that a given nondegenerate continuum X can be approached by a ray is measured only by cardinality, Corollary 1 is the best possible result. But, is it really the best from topological point of view? Theorem 1 shows that a huge collection of topologically distinct compactifications of $[1, \infty)$ with X as the remainder can be geometrically realized as the Cantor set of components of a certain compactum K. Since the decomposition of K into the set of its components is continuous, K and the collection of compactifications are topologically quite nice and simple. In this context it would be very interesting to know how bad and complicated the collection of all compactifications of $[1, \infty)$ with X as the remainder could be.

Question 1. Let X be a nondegenerate continuum. Is there a compactum K such that the following two conditions are satisfied?

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