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Proximate fixed point property and operations

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ABSTRACT

Klee introduced *the proximate fixed point property* for compacta which is stronger than fixed point property. We consider relations between proximate fixed point property of spaces being result of application of different operations to continua. As an application we show this property for products, cones, suspensions and joins of span 0 continua.

1. Introductory lemmas

AR means a compact absolute retract and ANR means a compact neighborhood retract. All spaces are considered to be subspaces of an AR compactum Q equipped with a metric ρ. The closed ε-neighborhood of X, i.e., $\{x \in Q : \rho(x, X) \leq \varepsilon\}$ of a set X will be denoted by X^ε. Notation $x \overset{\varepsilon}{=} y$ means that $\rho(x, y) \leq \varepsilon$. Let X, Y be compacta, and $\varepsilon > 0$. We say that a function $f : X \to Y$ is *ε-continuous* if there exist a number $\delta > 0$ such that if $\rho(x, x') < \delta$ then $\rho(f(x), f(x')) < \varepsilon$. Victor Klee in [5] introduced the following notion

Definition 1. A compactum X has the *proximate fixed point property* (pfpp in short) if for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that if $f : X \to X$ is δ-continuous then there exists a point $x \in X$ such that $x \overset{\varepsilon}{=} f(x)$.

The pfpp is a topological property and if a compactum has the pfpp then it has the fixed point property (Klee). A Warsaw circle, i.e., the set $\{(x, \sin \frac{3\pi}{2x}) : 0 < x \leq 1\} \cup \{(0, y) : -1 \leq y \leq 1\} \cup \{(\cos t, \sin t - 1) :$

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$-\pi \le t \le 0\} \subset \mathbb{R}^2$ represents an example of a continuum with the fixed point property but without the pfpp. If an ANR has the fixed point property then it has the pfpp as well (Klee). Jose Sanjurjo showed in [10] the following

Theorem 1 *(Sanjurjo). A compactum X has the proximate fixed point property if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if f is a mapping from X to X^δ then $x \overset{\varepsilon}{=} f(x)$ for a point $x \in X$.*

In the following we will use this rather than original definition of Klee. There are some simple but useful counterparts of well known fixed point theorems: Let us recall, that a mapping $f : X \to Y$ is *universal* if for every $g : X \to Y$ there exists $x \in X$ such that $f(x) = g(x)$ [4].

Proposition 1. *Let X be a continuum. If for every $\varepsilon > 0$ there exists a universal ε-mapping $f : X \to Y$, where Y_ε is an ANR then X has the pfpp.*

Proof. Let f be a universal $\varepsilon/2$-mapping of X onto Y – an ANR compactum Y. For some $\delta > 0$ we can extend it to an $\varepsilon/2$-mapping $g : X^\delta \to Y$. Now let $h : X \to X^\delta$. Because f is universal there exists a point $x \in X$ such that $f(x) = gh(x)$ and hence $f(x) \overset{\varepsilon}{=} x$. □

A consequence of Proposition 1 is:

Proposition 2. *Each ANR having fpp has pfpp as well.*

Proposition 3 *(Klee). Every retract Y of a continuum X with the pfpp has the pfpp as well.*

We call a mapping $f : X \to Y$ an *ε-shift* if $x \overset{\varepsilon}{=} f(x)$ for every $x \in X$

Proposition 4. *If for every number $\varepsilon > 0$ there exists an ε-shift of a continuum X into its subcontinuum X_ε with pfpp then X has pfpp.*

For us however, there will be useful a strengthening:

Proposition 5. *Let X be a continuum. If for every number $\varepsilon > 0$ there exists a subcontinuum X_ε with pfpp such that for every number $\xi > 0$ there exists an ε-shift $f_\varepsilon^\xi : X \to X_\varepsilon^\xi$ then X has pfpp.*

Proof. Let $\varepsilon > 0$ and let $\lambda = \varepsilon/3$. Choose $\eta > 0$ such that for every $\phi : X_\lambda \to X_\lambda^\eta$ there exists a point $x_0 \in X_\lambda$ such that $x_0 \overset{\lambda}{=} \phi(x_0)$. Consider a λ-shift $f : X \to X_\lambda^{\eta/2}$. It can be extended to a 2λ-shift $\tilde{f} : X^\delta \to X_\lambda^\eta$ for a number $\delta > 0$. Now, let $g : X \to X^\delta$. The composition $\tilde{f} \circ g$ maps X_λ into X_λ^η, hence we have a point $x_0 \in X_\lambda \subset X$ such that $x_0 \overset{\lambda}{=} \tilde{f} \circ g(x_0) \overset{2\lambda}{=} g(x_0)$. Thus $g(x_0) \overset{\varepsilon}{=} x_0$. □

An outer approximation is sometimes useful. One can easily show the following:

Proposition 6. *Let X be a continuum. If for every number $\varepsilon > 0$ there exists an ε-shift $f : X_\varepsilon \to X$, where X_ε is a continuum with pfpp such that $X \subset X_\varepsilon$ then the continuum X has pfpp.*

A countable product of compacta has projections being ε-shifts onto finite subproducts, for arbitrary $\varepsilon > 0$. Hence from Proposition 4 we have:

Proposition 7. *If for a family of continua X_i, $i = 1, 2, \dots$ each Cartesian product of finite subfamily has pfpp then the product $\prod_{i=1}^\infty X_i$ has pfpp as well.*