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Selective but not Ramsey

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ABSTRACT

We give a partial answer to the following question of Dobrinen: For a given topological Ramsey space $\mathcal{R}$, are the notions of selective for $\mathcal{R}$ and Ramsey for $\mathcal{R}$ equivalent? Every topological Ramsey space $\mathcal{R}$ has an associated notion of Ramsey ultrafilter for $\mathcal{R}$ and selective ultrafilter for $\mathcal{R}$ (see [1]). If $\mathcal{R}$ is taken to be the Ellentuck space then the two concepts reduce to the familiar notions of Ramsey and selective ultrafilters on ω; so by a well-known result of Kunen the two are equivalent. We give the first example of an ultrafilter on a topological Ramsey space that is selective but not Ramsey for the space.

We show that for the topological Ramsey space $\mathcal{R}_1$ from [2], the notions of selective for $\mathcal{R}_1$ and Ramsey for $\mathcal{R}_1$ are not equivalent. In particular, we prove that forcing with a closely related topological Ramsey space using almost-reduction, adjoins an ultrafilter that is selective but not Ramsey for $\mathcal{R}_1$.

1. Introduction

This paper is concerned with giving an example of a topological Ramsey space $\mathcal{R}$ and an ultrafilter that is selective for $\mathcal{R}$ but not Ramsey for $\mathcal{R}$. The first result of topological Ramsey theory was the infinite dimensional extension of the Ramsey theorem known as the Ellentuck theorem (see [3]). Ellentuck proved this theorem in order to give a proof of Silver's theorem stating that analytic sets have the Ramsey property. In order to state the Ellentuck theorem it is necessary to introduce the Ellentuck space.

We denote the infinite subsets of ω by $[\omega]^\omega$ and the finite subsets of ω by $[\omega]^{<\omega}$. If $B \in [\omega]^\omega$ and $\{b_0, b_1, b_2, \dots\}$ is its increasing enumeration, then for each $i < \omega$, we let $r_i(B)$ denote the set $\{b_0, b_1, b_2, \dots, b_{i-1}\}$ and call it the ith approximation of B. The *Ellentuck space* is the set $[\omega]^\omega$ of all infinite subsets of ω with the topology generated by the basic open sets,

$$[a, B] = \{A \in [\omega]^\omega : A \subseteq B \ \& \ (\exists i) r_i(A) = a\} \tag{1}$$

where $a \in [\omega]^{<\omega}$ and $B \in [\omega]^\omega$.

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Recall that a subset of a topological space is *nowhere dense* if its closure has empty interior and *meager* if it is the countable union of nowhere dense sets. A subset $\mathcal{X}$ of a topological space has the *Baire property* if and only if $\mathcal{X} = \mathcal{O}\Delta\mathcal{M}$ for some open set $\mathcal{O}$ and some meager set $\mathcal{M}$.

A subset $\mathcal{X}$ of the Ellentuck space is *Ramsey* if for every $\emptyset \neq [a, A]$, there is a $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$. The next theorem is the infinite-dimensional version of the Ramsey theorem.

Theorem 1 *(Ellentuck theorem). ([3]) Every subset of the Ellentuck space with the Baire property is Ramsey.*

Topological Ramsey spaces are spaces that have enough structure in common with the Ellentuck space that an abstract version of the Ellentuck theorem can be stated and proved. The Ellentuck space leads naturally to the notion of a selective ultrafilter on ω.

Definition 2. Let $\mathcal{U}$ be a nonprincipal ultrafilter on ω. If $i < \omega$ and A is an infinite subset of ω, *i.e.* in the Ellentuck space, then we let

$$A/i = A \setminus r_i(A). \tag{2}$$

$\mathcal{U}$ is *selective*, if for each decreasing sequence $A_0 \supseteq A_1 \supseteq \dots$ of members of $\mathcal{U}$ there exists $X = \{x_0, x_1, \dots\} \in \mathcal{U}$ enumerated in increasing order such that for all $i < \omega$,

$$A/i \subseteq A_i. \tag{3}$$

The next theorem, due to Kunen, characterizes selective ultrafilters as those which are minimal with respect to the Rudin–Keisler ordering.

Theorem 3. *([4]) Let $\mathcal{U}$ be an ultrafilter on ω. The following conditions are equivalent:*

1. *$\mathcal{U}$ is selective.*
2. *For each partition of the two-element subsets of ω into two parts, there is a set $X \in \mathcal{U}$ all of whose two-element subsets lie in one part of the partition.*
3. *Every function on ω is constant or one-to-one on some set in $\mathcal{U}$.*

An ultrafilter that satisfies the second item is called a *Ramsey* ultrafilter on ω. Generalizations of the previous theorem have been studied in many contexts. For example, the notions of selective coideal (see [5]) and semiselective coideals (see [6]) have been shown to also satisfy similar Ramsey properties. In [1], Mijares generalizes the notion of selective ultrafilter on ω to a notion of selective ultrafilter on an arbitrary topological Ramsey space $\mathcal{R}$. Mijares also generalizes the notion of Ramsey ultrafilter on ω to a notion of Ramsey ultrafilter for $\mathcal{R}$ and shows that if an ultrafilter is Ramsey for $\mathcal{R}$ then it is also selective for $\mathcal{R}$. If one takes $\mathcal{R}$ to be the Ellentuck space then the two generalizations reduce to the concepts of selective and Ramsey ultrafilter. The theorem of Kunen above shows that the notions of selective for the Ellentuck space and Ramsey for the Ellentuck space are equivalent. This leads to the following question asked by Dobrinen about the generalizations from selective and Ramsey to arbitrary topological Ramsey spaces.

Question 4. For a given topological Ramsey space $\mathcal{R}$, are the notions of selective for $\mathcal{R}$ and Ramsey for $\mathcal{R}$ equivalent?

Ramsey for $\mathcal{R}$ ultrafilters have also been studied by Dobrinen and Todorcevic in [2] and [7]. Motivated by Tukey classification problems, the authors develop a hierarchy of topological Ramsey spaces $\mathcal{R}_\alpha$, $\alpha \leq \omega_1$. Associated to each space $\mathcal{R}_\alpha$ is an ultrafilter $\mathcal{U}_\alpha$, which is Ramsey for $\mathcal{R}_\alpha$. The space $\mathcal{R}_0$ is taken to be the