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Smooth embeddings of the Long Line and other non-paracompact manifolds into locally convex spaces

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ABSTRACT

We show that every real finite dimensional Hausdorff (not necessarily paracompact, not necessarily second countable) C^r-manifold can be C^r-embedded into a weakly complete vector space, i.e. a locally convex space of the form $\mathbb{R}^I$ for an uncountable index set I and determine the minimal cardinality of I for which such an embedding is possible.

1. Introduction and statement of the results

We review the classical Theorem of Whitney (see e.g. [21, Theorem 1], [2, Theorem 6.3], or [1, Theorem 2.2]):

Theorem *(Whitney). Let M be a d-dimensional second countable Hausdorff C^r-manifold ($d \geq 1$). Then there exists a C^r-embedding into $\mathbb{R}^{2d}$.*

The conditions (Second Countability and the Hausdorff-property) are obviously also necessary, since every Euclidean space $\mathbb{R}^n$ is second countable and Hausdorff and so are all of its subsets. The dimension $2d$ in the Theorem is sharp in the sense that whenever d is a power of two, i.e. $d = 2^k$, there is a 2^k-dimensional second countable Hausdorff manifold which can not be embedded into $\mathbb{R}^{2\cdot 2^k - 1}$.

Unfortunately, not every Hausdorff C^r-manifold is second countable. Perhaps the easiest connected manifold where the second axiom of countability does not hold, is the *Long Line* and its relative the *Open Long*

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Ray (see e.g. [17, Example 3.2], [6, Example 1.5], or [12]). For the reader's convenience, we will recall the definition:

Definition 1.1 *(The Alexandroff Long Line).*

(a) Let ω_1 be the first uncountable ordinal. The product $\mathbb{L}_c^+ := \omega_1 \times [0,1[$, endowed with the lexicographical (total) order, becomes a topological space with the order topology, called the *Closed Long Ray*. This space is a connected (Hausdorff) one-dimensional topological manifold with boundary $\{(0,0)\}$.
(b) To obtain a manifold without boundary, one removes this boundary point. The resulting open set $\mathbb{L}^+ := \mathbb{L}_c^+ \setminus \{(0,0)\}$ is called the *Open Long Ray*.
(c) A different way to obtain a manifold without boundary is to consider two copies of the Closed Long Ray and glue them together at their boundary points. The resulting one-dimensional topological manifold $\mathbb{L}$ is called the *Long Line*.[1]

The spaces $\mathbb{L}, \mathbb{L}^+$, and $\mathbb{L}_c^+$ are locally metrizable but since countable subsets are always bounded, none of the three spaces is separable. So, in particular, they are not second countable. Since we are only considering manifolds without boundary in this paper, for us only the Open Long Ray and the Long Line are interesting.

It is known that there exist C^r-structures on $\mathbb{L}^+$ and on $\mathbb{L}$ for each $r \in \mathbb{N} \cup \{\infty\}$. They are however not unique up to diffeomorphism. For example, there are $2^{\aleph_1}$ pairwise non-diffeomorphic C^∞-structures on $\mathbb{L}$ (cf. [18]).

The Long Line and the Open Long Ray are by far not the only interesting examples of non-second countable manifolds. A famous two dimensional example (which has been known even before the Long Line) is the so called *Prüfer manifold*. We will not define it here but refer to [17, Example 3.7] or [19] for a definition.[2]

Since these manifolds fail to be second countable they cannot be embedded into a finite dimensional vector space. However, one can ask the question if it is possible to embed them into an infinite dimensional space. Of course, before answering this question, one has to say concretely what this should mean as there are different, non-equivalent notions of differential calculus in infinite dimensional spaces: We use the setting of Michal-Bastiani, based on Keller's C_c^r-calculus (see [7,15] and [16, Section I.2]). This setting allows us to work with C^r-maps between arbitrary locally convex topological vector spaces (*locally convex spaces* for short), as long as they are Hausdorff. A manifold modeled on a locally convex space can be defined via charts in the usual way, and there is a natural concept of a C^r-submanifold generalizing the concept in finite dimensional Euclidean space. Important examples of locally convex spaces are Hilbert spaces, Banach spaces, Fréchet spaces and infinite products of such spaces such as $\mathbb{R}^I$ for an arbitrary index set I. Since this differentiable calculus explicitly requires the Hausdorff property, we will not consider embeddings of non-Hausdorff manifolds, although there are interesting examples of those (occurring naturally as quotients of Hausdorff manifolds, e.g. leaf spaces of foliations etc.) Our first result is the following:

Theorem A. *Let $r \in \{1,2,\dots\} \cup \{\infty\}$. Let M be a finite dimensional Hausdorff C^r-manifold (not necessary second countable). Then there exists a set I such that M can be C^r-embedded into the locally convex space $\mathbb{R}^I$.*

A locally convex space of the type $\mathbb{R}^I$ is called a *weakly complete vector space* (see [3, Appendix C] or [10, Appendix 2]). These weakly complete spaces form a good generalization of finite dimensional vector spaces. The cardinality of the set I, sometimes called the *weakly complete dimension*, is a topological invariant of $\mathbb{R}^I$ (see Lemma 3.1). This gives rise to the following question: Given a finite dimensional C^r-manifold M, what

[1] Some authors use the term Long Line for what we call here Long Ray.
[2] See also [6, Examples 1.25 to 1.28] and [14, Examples 1.4.59 and 3.2.1] for generalizations of this concept called *Prüferization*.