Topology and its Applications 202 (2016) 80–105

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

# More Polish full groups

Alessandro Carderi [1], François Le Maître [*,2]

ARTICLE INFO

*Article history:*
Received 23 August 2015
Accepted 23 December 2015
Available online 12 January 2016

*Keywords:*
Measure-preserving actions
Orbit equivalence
Full groups
Polish groups

ABSTRACT

We associate to every action of a Polish group on a standard probability space a Polish group that we call the *orbit full group*. For discrete groups, we recover the well-known full groups of pmp equivalence relations equipped with the uniform topology. However, there are many new examples, such as orbit full groups associated to measure-preserving actions of locally compact groups. We also show that such full groups are complete invariants of orbit equivalence.

We give various characterizations of the existence of a dense conjugacy class for orbit full groups, and we show that the ergodic ones actually have a unique Polish group topology. Furthermore, we characterize ergodic full groups of countable pmp equivalence relations as those admitting non-trivial continuous character representations.



## 1. Introduction

One of the main problems in ergodic theory is to understand measure-preserving actions of a countable group $\Gamma$ on the standard probability space $(X, \mu)$ up to conjugacy. This problem turns out to be very complicated in many different ways, even for $\Gamma = \mathbb{Z}$. In this case, we are trying to understand the conjugacy classes of the group of automorphisms of the standard probability space, $\mathrm{Aut}(X, \mu)$.

The notion of orbit equivalence is much coarser than conjugacy. Two actions of countable groups $\Gamma$ and $\Lambda$ on the standard probability space $(X, \mu)$ are **orbit equivalent** if there is $S \in \mathrm{Aut}(X, \mu)$ such that for almost every $x \in X$, we have $S(\Gamma \cdot x) = \Lambda \cdot S(x)$. In other words, the orbit equivalence only keeps track of the partition of the space into orbits induced by $\Gamma$ and $\Lambda$ and not of the actions themselves. This is indeed weaker than conjugacy, for example a remarkable result of Ornstein and Weiss states that any two measure-preserving ergodic actions of any two amenable groups are orbit equivalent [33].

Orbit equivalence may be reformulated as follows. To any action of a countable group $\Gamma$ on the standard probability space $(X, \mu)$, we can associate the equivalence relation $\mathcal{R}_\Gamma$ on $X$ defined by $(x, y) \in \mathcal{R}_\Gamma$ if

* Corresponding author.
*E-mail addresses:* alessandro.carderi@gmail.com (A. Carderi), f.lemaitre@math.univ-paris-diderot.fr (F. Le Maître).

1 Research partially supported by Projet ANR-14-CE25-0004 GAMME.
2 Research supported by the Interuniversity Attraction Pole DYGEST and Projet ANR-14-CE25-0004 GAMME.

http://dx.doi.org/10.1016/j.topol.2015.12.065

there is $\gamma \in \Gamma$ such that $\gamma x = y$. It is then easy to observe two actions are orbit equivalent if and only if their equivalence relations are isomorphic up to measure zero. The equivalence relations arising in this way are called countable pmp (probability measure-preserving) equivalence relations. They have geometric and cohomological interpretations as well as fruitful relations with von Neumann algebras. We refer the interested reader to the survey of Gaboriau [15].

Another way of formulating orbit equivalence is due to Dye. Suppose that $\Gamma$ acts on the standard probability space $(X, \mu)$. The **full group** induced by the $\Gamma$-action is the set of all $T \in \mathrm{Aut}(X, \mu)$ such that for almost every $x \in X$, we have $T(x) \in \Gamma \cdot x$. This group still encodes orbit equivalence in the following sense: two actions are orbit equivalent if and only if their full groups are conjugate in $\mathrm{Aut}(X, \mu)$ and a theorem of Dye (see Theorem 3.27) implies that two actions are orbit equivalent if and only if their full groups are abstractly isomorphic.

As a consequence, one should be able to understand all the invariants of orbit equivalence in terms of full groups. This works well for ergodicity: an action is ergodic if and only if the associated full group is a simple group. Another example is given by aperiodicity: an action has no finite orbits if and only if its full group does not surject onto $\mathbb{Z}/2\mathbb{Z}$ (see [31, Thm. 1.14]).

In order to understand finer orbit equivalence invariants in terms of properties of full groups, one is led to introduce a Polish group topology on them. This topology is called the **uniform topology**, and it is induced by the **uniform metric** $d_u$ defined on $\mathrm{Aut}(X, \mu)$ by

$$d_u(T, S) := \mu\left(\{x \in X : \; Tx \neq Sx\}\right).$$

As an example, the uniform topology allows us to detect the amenability and the cost of an equivalence relation. Indeed, Giordano and Pestov proved in [17] that if $\Gamma$ acts freely on $(X, \mu)$, then $\Gamma$ is amenable if and only if the full group of the action is extremely amenable for the uniform topology. On the other hand, the second named author showed in [29] that the cost is related to the topological rank of a full group, that is the minimal number of elements needed to generate a (uniformly) dense subgroup.

In this work, we study full groups coming from actions of arbitrary Polish groups. Given a measure-preserving action of the Polish group $G$ on the standard probability space $(X, \mu)$, we define the **orbit full group** of the action exactly as before: it is the set of $T \in \mathrm{Aut}(X, \mu)$ such that for almost every $x \in X$, we have $T(x) \in G \cdot x$. We will denote this full group by $[\mathcal{R}_G]$ to remember that it is the full group of the equivalence relation induced by the $G$-action. We should warn the reader that our definition needs a concrete action of $G$ on $X$, and not just a morphism $G \to \mathrm{Aut}(X, \mu)$.

As in the case of discrete groups, in order to get a better understanding of orbit full groups we have to introduce a Polish topology on them. All the orbit full groups are closed for the uniform topology, but they are separable if and only if they arise as full groups of countable pmp equivalence relations. This, however, does not rule out the existence of a Polish topology on them. For instance a compact group acting on itself by translation generates the transitive equivalence relation and its orbit full group is $\mathrm{Aut}(X, \mu)$, which is a Polish group for the weak topology.

Our main goal is to define and study a Polish group topology on *any* orbit full group, which will not be the restriction to $[\mathcal{R}_G]$ of a topology on $\mathrm{Aut}(X, \mu)$ in general. We call this topology the **topology of convergence in measure**. When the action of $G$ on $X$ is free, this topology can be understood as follows: we associate to any element $T \in [\mathcal{R}_G]$ the function $f_T : X \to G$ uniquely defined by $T(x) = f_T(x) \cdot x$. Doing so, we embed $[\mathcal{R}_G]$ in the space of measurable functions from $X$ to $G$, and the Polish topology we will define coincides with the topology of convergence in measure.

**Theorem 1.** *(Theorem 3.17 and Theorem 4.7) Let $G$ be a Polish group acting in a measure-preserving Borel manner on a standard probability space $(X, \mu)$. Then the associated orbit full group*

$$[\mathcal{R}_G] = \{T \in \mathrm{Aut}(X, \mu) : \forall x \in X, T(x) \in G \cdot x\}$$