Topology and its Applications 202 (2016) 117–126

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Intrinsic shape of the chain recurrent set ☆

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A R T I C L E I N F O

Article history:
Received 1 March 2013
Received in revised form 7 January 2015
Available online 13 January 2016

MSC:
54H20
54C56
37B20
37B25

Keywords:
Chain recurrent set
Shape
Intrinsic shape
Attractor
Morse decomposition
Lyapunov function

A B S T R A C T

In this paper for the first time the shape of the chain recurrent set in a dynamical system is investigated. Following the already known theorem for shape of an attractor, we formulate a theorem about the shape of members of a Morse decomposition and the shape of the chain recurrent set.

One of the main applications of shape theory is in dynamical systems. A number of papers are written studying the shape properties of an attractor.

Recently, using the intrinsic shape (the approach to shape without use of external spaces) several results were obtained showing the advantages of intrinsic approach to shape in some situations. Applying intrinsic shape in dynamical systems seems very natural. In this paper we investigate the shape properties of the chain recurrent set introduced by Conley. The phase space X in which all dynamics takes place will be always a compact metric space.

☆ This paper was presented at the International Conference on Topology and its Applications, Skopje 2012 (ICTA 2012 Conference).

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http://dx.doi.org/10.1016/j.topol.2015.12.062
0166-8641/

1. Introduction to intrinsic shape for compact metric spaces

The first paper about intrinsic approach to shape is [4], where a shape morphism between compact metric spaces has been defined for the first time using the notion of $\mathcal{V}$-continuity (Definition 1.1). In the paper [11] the shape category of compact metric spaces has been defined using the notion of ε-continuity.

We will give a brief construction of the shape category using the intrinsic approach from [12] and [14]. The approach is based on the notion of $\mathcal{V}$-continuity (Definition 1.1) and a proximate sequence of functions (f_n), where $f_n : X \to Y$ is $\mathcal{V}_n$-continuous and $(\mathcal{V}_n)$ is a cofinal sequence of coverings (Definition 1.3).

In [12] it has been proved that the defined shape morphism coincides with the notion from [4] and in [8] that the obtained shape category is isomorphic with the category from [11]. In [7] it has been proved that the notion of intrinsic shape in [12] is equivalent to the Borsuk's original notion of shape [1].

Let's start with some of the basic definitions:

For collections $\mathcal{U}$ and $\mathcal{V}$ of subsets of X, we denote $\mathcal{U}$ refines $\mathcal{V}$ by $\mathcal{U} \prec \mathcal{V}$, i.e. each $U \in \mathcal{U}$ is contained in some $V \in \mathcal{V}$. By covering we understand a covering consisting of open sets.

Definition 1.1. Suppose $\mathcal{V}$ is a finite covering of Y. A function $f : X \to Y$ is $\mathcal{V}$-continuous at a point $x \in X$, if there exists a neighborhood U_x of x and $V \in \mathcal{V}$, such that:

$$f(U_x) \subseteq V.$$

A function $f : X \to Y$ is $\mathcal{V}$-continuous, if it is $\mathcal{V}$-continuous at every point $x \in X$. In this case, the family of all U_x, form a covering of X.

According to this, $f : X \to Y$ is $\mathcal{V}$-continuous if there exists a finite covering $\mathcal{U}$ of X, such that for any $x \in X$, there exists $U \in \mathcal{U}$, a neighborhood of x, and $V \in \mathcal{V}$ such that $f(U) \subseteq V$. We denote shortly: there exists $\mathcal{U}$, such that $f(\mathcal{U}) \prec \mathcal{V}$.

Let $\mathcal{V}$ be a finite covering of Y and $V \in \mathcal{V}$. The open set $\mathrm{st}(V)$ (star of V) is the union of all $W \in \mathcal{V}$ such that $W \cap V \neq \emptyset$. We form a new covering of Y, $\mathrm{st}(\mathcal{V}) = \{\mathrm{st}(V) | V \in \mathcal{V}\}$.

Definition 1.2. The functions $f, g : X \to Y$ are $\mathcal{V}$-homotopic, if there exists a function $F : X \times I \to Y$ such that:

i) $F : X \times I \to Y$ is $\mathrm{st}(\mathcal{V})$-continuous,
ii) $F : X \times I \to Y$ is $\mathcal{V}$-continuous at all points of $X \times \partial I$,
iii) $F(x, 0) = f(x)$, $F(x, 1) = g(x)$.

The relation of homotopy is an equivalence relation and is denoted by $f \overset{\mathcal{V}}{\simeq} g$.

Note 1.1. The conditions *i)* and *ii)* in the definition of homotopy allow concatenation of homotopies and cannot be replaced by the expected condition $F : X \times I \to Y$ to be $\mathcal{V}$-continuous. This is explained by the next proposition from [12].

Proposition 1.1. *Suppose $\mathcal{V}$ is a finite covering of Y, $X = X_1 \cup X_2$, X_i closed, $i = 1, 2$ and $f_i : X_i \to Y$, $\mathcal{V}$-continuous functions, $i = 1, 2$ such that $f_1(x) = f_2(x)$, for all $x \in X_1 \cap X_2$. We define a function by*

$$f(x) = f_i(x), \ \text{for } x \in X_i, \ i = 1, 2,$$