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Integral positive (negative) quandle cocycle invariants are trivial for knots ☆

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A R T I C L E I N F O

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In this note we prove that for any finite quandle X and any 2-cocycle $\phi \in Z^2_{Q\pm}(X;\mathbb{Z})$, the cocycle invariant $\Phi^{\pm}_{\phi}(K)$ is trivial for all knots K.

1. Introduction

A quandle [9,12] is a set with a binary operation which satisfies some axioms motivated by the three Reidemeister moves. Similar to the knot group, for every knot $K \subset S^3$ one can define the knot quandle $Q(K)$ [9]. It is well known that $Q(K)$ is a powerful knot invariant. It characterizes the knot K up to reverse mirror image. However in general the knot quandle is not easy to deal with, therefore it is more convenient to count the homomorphisms from $Q(K)$ to a fixed finite quandle, known as the quandle coloring invariant. More sophisticated invariants of this sort were introduced via quandle cohomology in [3]. More precisely, as a modification of the rack cohomology theory [6,7], J.S. Carter et al. constructed the quandle cohomology theory, then they showed that with a given quandle 2-cocycle (3-cocycle) one can generalize the quandle coloring invariant to a state-sum invariant for knots (knotted surfaces). In particular when the 2-cocycle is a coboundary this state-sum invariant reduces to the quandle coloring invariant. In [4] we consider another boundary map of the chain complex and introduce the positive quandle cohomology theory. Similar to the quandle cohomology theory, with a given positive quandle 2-cocycle and 3-cocycle we can also define a

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state-sum invariant for knots and knotted surfaces respectively. Besides of trivial quandles, it was proved that if we choose a dihedral quandle then the quandle cocycle invariants and positive quandle cocycle invariants are both trivial (i.e. reduce to the quandle coloring invariant) for knots [3,4]. The main result of this note is an extension of this fact.

Theorem 1.1. *Let X be a finite quandle and $\phi \in Z^2_{Q-}(X;\mathbb{Z})$, then for any knot K the cocycle invariant $\Phi^-_\phi(K)$ is trivial.*

Theorem 1.2. *Let X be a finite quandle and $\phi \in Z^2_{Q+}(X;\mathbb{Z})$, then for any knot K the cocycle invariant $\Phi^+_\phi(K)$ is trivial.*

We remark that if we replace the coefficient group $\mathbb{Z}$ with some other coefficient groups, for example $\mathbb{Z}_2$, then the positive (negative) quandle cocycle invariants need not to be trivial. The readers are referred to [3] for some nontrivial examples. On the other hand, if we replace the knot K with a link which has at least two components, then even using the trivial quandle with two elements one can prove that the negative quandle cocycle invariants contain the information of pairwise linking numbers. For the positive quandle cocycle invariant, one can even use it to distinguish the Borromean ring from the 3-component trivial link. See [3] and [4] for more details.

The remainder of this note is arranged as follows. Section 2 contains the basic definitions of quandle, positive (negative) quandle (co)homology groups and the cocycle invariants. Section 3 and section 4 are devoted to the proof of Theorem 1.1 and Theorem 1.2 respectively.

2. Quandle homology and cocycle invariants

A *quandle* $(X, *)$, is a set X with a binary operation $(a, b) \to a * b$ satisfying the following axioms:

1. For any $a \in X$, $a * a = a$.
2. For any $b, c \in X$, there exists a unique $a \in X$ such that $a * b = c$.
3. For any $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.

Note that with a given binary operation $*$ we can define the dual operation $*^{-1}$ by

$$c *^{-1} b = a \text{ if } a * b = c.$$

It is easy to observe that $(X, *^{-1})$ is also a quandle, called the *dual quandle* of $(X, *)$. For convenience we simply denote $(X, *)$ by X. If X satisfies the second and the third axioms, then we name it a *rack*. Here we list some basic examples of quandle:

- Let $T_n = \{a_1, \cdots, a_n\}$ and $a_i * a_j = a_i$, we say T_n is the trivial quandle with n elements;
- Let $D_n = \{0, 1, \cdots, n-1\}$ and $i * j = 2j - i \pmod n$, we say D_n is the dihedral quandle of order n;
- Let X be a conjugacy class of a group G and $a * b = b^{-1}ab$, we say X is a conjugation quandle.

Let X be a quandle and a, b two elements of X. We say a and b are of the same *orbit* if b can be obtained from a by some right translations, i.e. there exist some elements $\{a_1, \cdots, a_n\} \subset X$ such that

$$b = (\cdots((a *^{\varepsilon_1} a_1) *^{\varepsilon_2} a_2) \cdots) *^{\varepsilon_n} a_n,$$

where $\varepsilon_i \in \{\pm 1\}$. The orbit set of X is denoted by $\mathrm{Orb}(X)$, and the orbit that contains a is denoted by $\mathrm{Orb}(a)$. Obviously $\mathrm{Orb}(a)$ is a subquandle of X. If X has only one orbit then we say X is *connected*.