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Euclidean symmetry of closed surfaces immersed in 3-space

Undine Leopold [a], Thomas W. Tucker [b,*]

[a] Fakultät für Mathematik, Technische Universität Chemnitz, 09107 Chemnitz, Germany
[b] Mathematics Department, Colgate University (Emeritus), Hamilton, NY 13346, USA

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ABSTRACT

Given a finite group G of orientation-preserving isometries of euclidean 3-space E^3 and a closed surface S, an immersion $f : S \to E^3$ is in G-general position if $f(S)$ is invariant under G, points of S have disk neighborhoods whose images are in general position, and no singular points of $f(S)$ lie on an axis of rotation of G. For such an immersion, there is an induced action of G on S whose Riemann–Hurwitz equation satisfies certain natural restrictions. We classify which restricted Riemann–Hurwitz equations are realized by a G-general position immersion of S. This generalizes work by various authors on euclidean symmetry of closed surfaces embedded in E^3. The analysis involves a detailed study of immersions of the quotient surface S/G in the orbifold E^3/G.

Given a group G of orientation-preserving isometries of euclidean 3-space E^3 and a closed surface S, a continuous map $f : S \to E^3$ is a *G-general position immersion* if every point in S has a disk neighborhood on which f is an embedding, the images of these disks are in general position in E^3, no singular point of f lies on an axis of rotation of G, and $f(S)$ is invariant under G. The action of G on $f(S)$ induces a pseudo-free (isolated fixed points) action of G on S with regular branched covering $p : S \to T$. The Riemann–Hurwitz equation relates the Euler characteristic $\chi(S)$ to $\chi(T)$ and the number of branch points of each order.

In this paper, we classify which Riemann–Hurwitz equations for a pseudo-free action on S of a finite group G of orientation-preserving isometries of E^3 are realizable by a G-general immersion of S in E^3. The possibilities for G are the cyclic group C_n for a n-fold rotation, the dihedral group D_n for symmetry of the n-prism ($n > 1$), and the groups A_4, S_4, and A_5 for the symmetry of the tetrahedron, cube/octahedron, and dodecahedron/icosahedron. The restrictions on the associated Riemann–Hurwitz equation are that the branch points all have full order and the parities of the number of branch points of each type are the same.

* Corresponding author.
E-mail addresses: undine.leopold@mathematik.tu-chemnitz.de (U. Leopold), ttucker@colgate.edu (T.W. Tucker).

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In the case where S is orientable and T is not, G must have an index two subgroup containing all branching and $\chi(T)$ must be even. We show that these restricted Riemann–Hurwitz equations are realizable, with exceptions only when $\chi(T) \geq -2$ and there is insufficient branching; the details for these exceptions are complicated.

This question has been considered for embedded (and hence orientable) surfaces, first with G cyclic [18] and more recently for other finite groups [19] (inspired by a question by Bojan Mohar about the sculpture [8]). An erroneous statement in [19] is corrected in this paper. The original context was conformal automorphisms of Riemann surfaces and has been generalized to anti-conformal automorphisms [5,7,12] and bordered surfaces [4]. There seems to be no study of the symmetry of immersed closed surfaces.

The organization of this paper is as follows. Section 1 describes the orbifolds for the group G. Section 2 outlines our approach using immersions of the quotient surface T in the orbifold for G. It also presents the three restrictions that must be satisfied by a realizable Riemann–Hurwitz equation. Section 3 relates the restrictions to two conditions on fundamental groups. Section 4 gives the theorems saying, for each G, which restricted Riemann–Hurwitz equations are realizable. Section 5 revisits [19] to list which surfaces have a G-general immersion, with no specific Riemann–Hurwitz equation. Section 6 discusses possible generalizations.

1. The orbifolds of E^3

Our approach is to take a finite group G of orientation-preserving isometries of euclidean 3-space E^3 and look at its *orbifold* E^3/G, namely the quotient space obtained by identifying each orbit of G to a single point. The possibilities for G are easy to describe, since the barycenter of any orbit must be fixed, so we can assume at the outset that G leaves, say, the origin fixed, and therefore leaves invariant the sphere of radius d, for each choice of d. The orientation-preserving finite groups acting on the sphere are well known: the rotation groups for n-prisms and platonic solids and their subgroups.

We can visualize the quotient map $q : E^3 \to E^3/G$ as acting on the set of concentric spheres centered at the origin (having if we wish the appropriate prismatic or platonic shape). Restricted to each such sphere, q is a regular branched covering of the sphere with two (if $G = C_n$) or three branch points. Thus q itself is a regular branched covering of E^3, with branching on one, two or three collections of rotation axes, whose union we denote X.

We consider now the geometry of $Y = q(X)$ in E^3/G for each of the possibilities for G. For the cases where G is not cyclic, we give a presentation for G where u, v represent rotations around different axes of X. The comments about index two subgroups are for later use.

Rotation Here G is the cyclic group generated by a rotation about a line X through the origin. Then Y is again a line through the origin.

The n-prism The group is the dihedral group D_n. The descriptions are slightly different depending on the parity of n. In both cases, the n-fold axis is folded in half. When n is even, there are two types of 180 degree rotation: those going through the midpoints of vertical edges and those going through the centers of the side faces. Each gets folded in half because the 2-fold axes form pairs of perpendicular axes. Thus Y consists of two 2-fold rays and one n-fold ray emanating from the origin. One can think of the n-fold ray as the z axis, while the two 2-fold rays come from axes in the xy-plane making an angle of π/n. When n is odd, there is no symmetry exchanging the halves of a 2-fold axis, but the group G acts transitively on the 2-fold axes. Thus Y consists of one n-fold ray and a 2-fold line, perpendicular to each other at the origin. In both cases, the group G has the presentation $\langle u, v : u^2 = v^2 = (uv)^n = 1\rangle$. When n is odd $\langle uv\rangle$ is the only index two subgroup, but when n is even, $\langle u, (uv)^2\rangle$ and $\langle v, (uv)^2\rangle$ also have index two.