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Extreme entropy versus growth rates of periodic orbits in equivalent flows

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ABSTRACT

In this paper, we prove that there exists a pair of equivalent flows with fixed points such that one of which has 0 topological entropy and ∞ growth rate of periodic orbits but the other has ∞ topological entropy and 0 growth rate of periodic orbits.

1. Introduction

Two flows on compact metric spaces are equivalent if there exists a homeomorphism of the spaces that sends each orbit of one flow onto an orbit of the other flow while preserving the time orientation. The topological entropy and the growth rate of periodic orbits, which coincides in Anosov diffeomorphisms or Anosov flows [1], are both measures of complexity of a system. It is known that a pair of conjugate homeomorphisms share the same topological entropy. It is obvious that a pair of conjugated homeomorphisms share the periods for corresponding periodic orbits and thus share the growth rate of periodic orbits. However, a pair of equivalent flows might share neither entropy nor the growth rate of periodic orbits, due to time reparameterization between the two flows. In this paper, we will construct a pair of equivalent flows with

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fixed points such that one of which has 0 topological entropy and ∞ growth rate of periodic orbits but the other has ∞ topological entropy and 0 growth rate of periodic orbits.

As to the topological entropy, the authors have already showed in [8], by constructing examples, that the flow with 0 topological entropy and the flow with ∞ topological entropy exist simultaneously in certain class of equivalent flows with fixed points. As to the growth rate of periodic orbits, they showed the extreme phenomenon in [9]. Nevertheless, the conclusion in the present paper is not a corollary of them. In fact, topological entropy and the growth rate of periodic orbits are generally different measures of complexity of a flow (although they coincide in an Anosov flow). Moreover, we show the extreme difference: for the pair of equivalent flows, one of which has 0 topological entropy and ∞ growth rate of periodic orbits but another has ∞ topological entropy and 0 growth rate of periodic orbits.

We point out that there must be a fixed point in each flow, since the extreme values of either entropy or the growth rate of periodic orbits are invariant in non-singular equivalent flows [7,9]. Fixed points do play an important role in the complexity of flows. The decelerations of the flows near the fixed point cause the divergence of entropy and growth rate of periodic orbits.

Let M be a compact metric space. Let $\phi : M\times\mathbb{R} \to M$ be a flow, i.e., a continuous map satisfying $\phi(x, 0) = x$ and $\phi(\phi(x, s), t) = \phi(x, s + t)$, $\forall x \in M$ and $s,\ t \in \mathbb{R}$. For given $t \in \mathbb{R}$, we denote the homeomorphism $\phi_t(\cdot) : M \to M$ by $\phi_t(x) = \phi(x, t)$. Given $A \in \mathbb{R}^+$, we also denote *the number of periodic orbits of at most A-period* by $\pi(\phi, A)$, i.e.,

$$\pi(\phi, A) = \max\{\sharp\{orb(x)|x \in M,\ \ \phi(x, u) = x \text{ for some } u \leq A\}, 1\}.$$

Let

$$p(\phi, A) = \frac{1}{A}\log \pi(\phi, A) \text{ and } p(\phi) = \limsup_{A\to\infty} p(\phi, A).$$

We call $p(\phi)$ the *growth rate of periodic orbits* for ϕ. We denote by $h(\phi)$ the topological entropy of ϕ. That is to say, $h(\phi) = h(\phi_1)$. Now we state our main theorem.

Theorem 1.1. *There exists a pair of equivalent flows $\Phi : M \times \mathbb{R} \to M$ and $\Psi : W \times \mathbb{R} \to W$ on compact metric spaces M and W, respectively, such that*

1. $p(\Phi) = \infty$, $h(\Phi) = 0$;
2. $p(\Psi) = 0$, $h(\Psi) = \infty$.

Let us note that the situation of Theorem 1.1 only exists in the flows with fixed points. Actually, both extreme topological entropy and extreme growth rate of periodic orbits are invariant in equivalent flows without fixed points [7,9].

2. Growth rate of the flows

Theorem 2.1. *There exists a pair of equivalent flows $\phi : M' \times \mathbb{R} \to M'$ and $\psi : W' \times \mathbb{R} \to W'$ with fixed points on compact metric spaces M' and W', respectively, such that*

1. $p(\phi) = \infty$;
2. $p(\psi) = 0$.

Proof. The existence of such equivalent flows is guaranteed by [9]. For convenience to readers we briefly recall it as follows.