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Some remarks on open covers and selection principles using ideals ☆

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ABSTRACT

In this paper we follow the line of recent works of Das [4,5,7] (see also [3] for similar investigation), where a more general approach was made to study certain results on open covers and selection principles by using the notion of ideals and ideal convergence, which automatically extends similar classical results (where finite sets are used), and also recent statistical variants studied by Di Maio and Kočinac [10]. Here we further introduce the notions of $c\omega$-covers and $\mathcal{I}$-large covers which extend the notions of ω-covers and large covers in a topological space. We then study the $\mathcal{I}$-groupability of different open covers and some of its consequences.

1. Introduction

In [21] (see also [13]) M. Scheepers began a systematic study of selection principles in topology and their relations to game theory and Ramsey theory. After that many papers on this topic appeared in

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the literature. Various topological properties have been defined or characterized in terms of the classical selection principles defined in the next section. In [10], the idea of statistical convergence was applied to define selection principles using actually the ideal of asymptotic density zero sets of natural numbers. This idea was extended to arbitrary ideals on $\mathbb{N}$ in the papers [5] and [4] and was continued in [7]. In this paper we continue this sort of investigation. Also it should be noted that a similar approach was made using filters and in particular semi-filters. In [20] the dual approach of looking at filters instead of ideals is explored (see also [1,3,25,26]).

Our topological terminology and notation are as in the book [11]. For undefined notions regarding selection principles in topological spaces we refer the reader to the survey papers [15,16,23,24] and references therein.

2. Preliminaries

Throughout the paper (X, τ) stands for a Hausdorff topological space. For two non-empty classes of sets $\mathcal{A}$ and $\mathcal{B}$ of an infinite set S, following [21,13], we define:

$\mathcal{S}_1(\mathcal{A}, \mathcal{B})$: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$, there is a sequence $(b_n : n \in \mathbb{N})$ such that $b_n \in A_n$ for each n and $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$.

$\mathcal{S}_{fin}(\mathcal{A}, \mathcal{B})$: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$, there is a sequence $(B_n : n \in \mathbb{N})$ of finite (possibly empty) sets such that $B_n \subset A_n$ for each n and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

There are infinitely long games corresponding to these selection principles (see [21,13,22]).

$\mathcal{G}_1(\mathcal{A}, \mathcal{B})$ denotes the game for two players, ONE and TWO, who play a round for each positive integer n. In the n-th round ONE chooses a set A_n from $\mathcal{A}$ and TWO responds by choosing an element $b_n \in A_n$. TWO wins the play $(A_1, b_1; ...; A_n, b_n; ...)$ if $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$; otherwise ONE wins.

$\mathcal{G}_{fin}(\mathcal{A}, \mathcal{B})$ denotes the game where in the n-th round ONE chooses a set A_n from $\mathcal{A}$ and TWO responds by choosing a finite (possibly empty) set $B_n \subset A_n$. TWO wins the play $(A_1, B_1; ..., A_n, B_n; ...)$ if $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$; otherwise ONE wins.

We now recall some classes of open covers which we will use throughout the paper. Let $\mathcal{O}$ denote the collection of all open covers of the topological space X. An open cover $\mathcal{U}$ of X is an *ω-cover* if X is not a member of $\mathcal{U}$ and for each finite subset F of X there is a $U \in \mathcal{U}$ such that $F \subset U$ [12]. We use the symbol Ω to denote the set of all ω-covers. An open cover of X is said to be *large* if for each point there are infinitely many sets in the cover containing that point. The symbol Λ will be used to denote the set of all large covers [21,13,17]. Note that there is generally no restriction on the cardinality of an ω cover or a large cover. But in practice, most often they are assumed to be countable covers (see for example [21,13,17]).

Let S be an infinite set and $\mathcal{A}$ be a family of subsets of S. An element A of $\mathcal{A}$ is said to be *groupable* if there is a partition $A = \bigcup_{n\in\mathbb{N}} A_n$ into pairwise disjoint finite sets such that for each infinite set N of positive integers, also $\bigcup\{A_n : n \in N\} \in \mathcal{A}$. The symbol $\mathcal{A}^{gp}$ will be used to denote the set of groupable elements of $\mathcal{A}$ [14,17]. For a topological space X and for $x \in X$, Ω_x denotes the set $\{A \subset X \setminus \{x\} : x \in \overline{A}\}$. A topological space X is called an *ϵ-space* if each ω-cover of X has a countable subset which is an ω-cover [12].

Throughout the paper $\mathbb{N}$ will denote the set of all positive integers. A family $\mathcal{I} \subset 2^Y$ of subsets of a non-empty set Y is said to be an *ideal* in Y if (i) A, $B \in \mathcal{I}$ implies $A \bigcup B \in \mathcal{I}$ (ii) $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$, while an *admissible ideal* $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$. Such ideals are also called *free ideals*. If $\mathcal{I}$ is a proper ideal in Y (i.e. $Y \notin \mathcal{I}$, $\mathcal{I} \neq \{\emptyset\}$), then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset Y$: there exists $A \in \mathcal{I} : M = Y \setminus A\}$ is a *filter* in Y. It is called the filter associated with the ideal $\mathcal{I}$. Throughout the paper $\mathcal{I}$ will stand for a proper ideal of $\mathbb{N}$. We denote the ideal of all finite subsets of $\mathbb{N}$ by $\mathcal{I}_{fin}$.

In this connection it can be mentioned that Kostyrko et al. [18] considered arbitrary ideals $\mathcal{I}$ on $\mathbb{N}$ and defined the notion of $\mathcal{I}$-convergence of sequences extending the idea of statistical convergence. Following