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Coarse metric approximation ☆

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A B S T R A C T

We generalize the concept of coarsely n-to-1 maps by introducing coarsely finite-to-1 maps and show that the new concept is a defining property of large scale finitistic spaces. Furthermore, we introduce an approximation of large scale (also called coarse) spaces by metric spaces, called the coarse metric approximation, and develop the corresponding theory. The mentioned approximation is a tool which allows us to generalize statements about coarse properties of metric spaces to equivalent statements about general large scale spaces. As a result we generalize the coarse versions of the Dimension Raising Theorem, Finite-To-One Mapping Theorem and the obtained characterization of large scale finitistic spaces to the case of general coarse structures.

1. Introduction

Approximations of spaces by a parameterized family of simplicial complexes are a well established tool and have applications in a variety of places. These include:

- Anti-Čech approximations introduced by Higson and Roe in [10], used in Roe's [13] and Dranishnikov's [7] (approximations for large parameters);
- the pipeline of the topological data analysis using persistent homology (approximations for intermediate parameters);
- the classical Čech approximation, which forms a basis of the shape theory (approximations for small parameters).

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The fundamental machinery in coarse approximation include Vietoris–Rips complexes or Čech complexes P_r with $r \to \infty$ and nerves of increasing families of uniformly bounded covers. For a similar approach see [6]. It is of fundamental importance to understand the geometry of uniformly bounded covers as they increase in size and this is the fundamental idea behind coarse geometry.

The motivation behind this work stems from several works. One major source of ideas comes out of [12] which first explores the dual notion of n-to-1 functions in the large scale. Another comes from [2] which explores the duality between large and small scale geometry and concurrently introduces direct metric approximations. The philosophy behind this paper is to continue to explore dualizations of small scale geometry to large scale and to use coarsely n-to-1 functions as a setting to show the utility of metric approximations.

In the first part of the paper we introduce the notion of coarsely finite-to-1 maps. It turns out that coarsely n-to-1 maps, as defined and studied in [12], represent the class of maps, for which the coarse versions of the Dimension Raising Theorem and Finite-To-One Mapping Theorem exist. In particular, the later theorem provides a characterization of asymptotic dimension in terms of such maps. We prove analogous results for coarsely finite-to-1 maps and large scale finitistic spaces. In particular, we provide the characterization of large scale finitistic spaces in terms of coarsely finite-to-1.

In the second part of the paper we describe the coarse metric approximation. It is well known that all uniform spaces are naturally built from metric spaces. Given a uniform space X, one takes the set D all metrics d on X such that the identity function $X \to (X, d)$ is uniformly continuous. The set of uniform covers on X is the smallest possible family of covers such that $X \to (X, d)$ is uniformly continuous for all $d \in D$. Instead of starting with a uniform structure, all one needs is a collection D of metrics d on a set X that is directed in the following sense: $d \geq d'$ if $id\colon (X, d) \to (X, d')$ is uniformly continuous. The union of all the uniform structures (X, d) is the original uniform structure. We show how to dualize this argument and demonstrate that every large scale structure can be constructed that way, just as it is the case of small scale structures (uniform spaces). The resulting tool, called the coarse metric approximation, is then applied in order to obtain the generalizations of coarse versions of the Dimension Raising Theorem, Finite-To-One Mapping Theorem and the obtained characterization of large scale finitistic spaces to the case of general coarse structures (see Corollaries 4.17, 4.19 and 4.20).

2. Preliminaries

Let X be a set and let $\mathcal{U}$ and $\mathcal{V}$ be covers of X. The **star of a subset** $A \subset X$ **against** $\mathcal{U}$ is given by $st(A, \mathcal{U}) = \cup\{U \in \mathcal{U} \mid U \cap A \neq \emptyset\}$. The **star of** $\mathcal{U}$ **with respect to** $\mathcal{V}$ is $st(\mathcal{U}, \mathcal{V}) = \{st(U, \mathcal{V}) \mid U \in \mathcal{U}\}$. Motivation for starring comes from metric spaces because the star of a subset of a metric space against the cover of the space by all r-bounded sets (set A is r-bounded if its diameter $\operatorname{diam}(A) = \sup_{x,y \in A} d(x, y)$ is at most r) just gives the r-neighborhood of the original subset. One typically thinks of starring a subset A against a cover $\mathcal{U}$ as thickening up A by $\mathcal{U}$. The **multiplicity** of a covering $\mathcal{U}$ is the maximum number of elements of $\mathcal{U}$ whose intersection is nontrivial and is ∞ if no such maximum exists. We say that a cover $\mathcal{U}$ is a **coarsening** of another cover $\mathcal{V}$ (and that $\mathcal{V}$ is a **refinement** of $\mathcal{U}$) if every element of $\mathcal{V}$ is contained in some element of $\mathcal{U}$ and we denote this situation by $\mathcal{U} \geq \mathcal{V}$.

The following definition is motivated by the coarse structure of metric spaces. One intuitively thinks of a uniformly bounded cover as if it were a cover of a metric space by R-balls. So the following definition just captures the essence of the covers of metric spaces by R-balls where R ranges over all nonnegative reals.

Definition 2.1. ([8]) A **Large Scale Structure(ls-structure)** on a set X is a nonempty set of families $\mathcal{LSS}$ of subsets of X satisfying

1) $\mathcal{B}_1 \in \mathcal{LSS}$ implies $\mathcal{B}_2 \in \mathcal{LSS}$ if each nonsingleton element of $\mathcal{B}_2$ is contained in some element of $\mathcal{B}_1$.
2) If $\mathcal{B}_1$, $\mathcal{B}_2 \in \mathcal{LSS}$ then $st(\mathcal{B}_1, \mathcal{B}_2) \in \mathcal{LSS}$.