Topology and its Applications 202 (2016) 216–238

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Recognition of plane-to-plane map-germs

Yutaro Kabata

Department of Mathematics, Graduate School of Science, Hokkaido University, Sapporo 060-0810, Japan

A R T I C L E   I N F O

Article history:
Received 7 November 2014
Received in revised form 9 September 2015
Accepted 13 January 2016

MSC:
57R45
53A05
53A15

Keywords:
$\mathcal{A}$-Classification of map-germs
Recognition problem
Projection

A B S T R A C T

We present a complete set of criteria for determining $\mathcal{A}$-types of plane-to-plane map-germs of corank one with $\mathcal{A}$-codimension ≤ 6, which provides a new insight into the $\mathcal{A}$-classification theory from the viewpoint of recognition problem. As an application to generic differential geometry, we discuss about projections of smooth surfaces in 3-space.

1. Introduction

We revisit the $\mathcal{A}$-classification of local singularities of plane-to-plane maps. Here $\mathcal{A}$ denotes the group of diffeomorphism germs of source and target planes preserving the origin. The classification has been achieved by J.H. Rieger, M.A.S. Ruas [15,16,18] – for instance, Table 1 below shows the list of all corank one map-germs with $\mathcal{A}$-codimension ≤ 6. When we apply the classification to some specific geometric situation, it often becomes a cumbersome task to detect which $\mathcal{A}$-type a given map-germ belongs to, which is referred to as "$\mathcal{A}$-recognition problem" (cf. [6]). In fact, Rieger's algorithm frequently uses Mather's Lemma to reduce the jet to some nicer form, at which the coordinate changes are not explicitly given (dotted lines in the recognition trees Figs. 1–5 in [15] indicate such processes). To fill up the process is not easy: the task is essentially related to deeper understanding on a filtered structure of the $\mathcal{A}$-tangent space of the germ, as T. Gaffney pointed out in an earlier work [6].

In this paper, we present a complete set of criteria for detecting $\mathcal{A}$-types of corank one germs with $\mathcal{A}$-codimension ≤ 6 (Theorem 3.1). That is a useful package consisting of two-phased criteria (Table 3 and

E-mail address: kabata@math.sci.hokudai.ac.jp.

http://dx.doi.org/10.1016/j.topol.2016.01.011

Table 4), which would easily be implemented in computer. The first one is about geometric conditions on 'specified jets' for *topological* $\mathcal{A}$*-types* in terms of *intrinsic derivatives* [23,19,20,14,9], and the second is about algebraic conditions on *Taylor coefficients* of germs with each specified jet, which are obtained by describing explicitly all the required coordinate changes of source and target of map-germs which are hidden in the classification process (Propositions 3.3, 3.5, 3.6, and 3.8).

For example, look at the cases of the butterfly $(x, xy + y^5 \pm y^7)$ and the elder butterfly $(x, xy + y^5)$, which are combined into a single topological $\mathcal{A}$-type. Suppose that a map-germ $f = (f_1, f_2) : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ with corank one is given. Put $\lambda(x, y) := \frac{\partial(f_1,f_2)}{\partial(x,y)}$, and take an arbitrary vector field $\eta := \eta_1(x, y)\frac{\partial}{\partial x} + \eta_2(x, y)\frac{\partial}{\partial y}$ near the origin of the source space so that η spans $\ker df$ on $\lambda = 0$ (Saji [19]). Denote $\eta^k g := \eta(\eta^{k-1} g)$. We show that the corresponding weighted homogeneous *specified jet* (see section 2.2) is characterized in terms of λ and η:

$$j^5 f(0) \sim_{\mathcal{A}^5} (x, xy + y^5) \iff \begin{cases} d\lambda(0) \neq 0, \\ \eta\lambda(0) = \eta^2\lambda(0) = \eta^3\lambda(0) = 0, \quad \eta^4\lambda(0) \neq 0 \end{cases}$$

Notice that the condition on the right hand side does not depend on the choices of local coordinates and the null vector field η. The subtle difference between these (C^∞-)$\mathcal{A}$-types is expressed by the following *Taylor coefficients condition*: If we write $f = (x, xy + y^5 + \sum_{i+j\geq 6} a_{ij} x^i y^j)$, then

$$f \sim_{\mathcal{A}} (x, xy + y^5 \pm y^7) \iff a_{07} - \tfrac{5}{8} a_{06}^2 \neq 0,$$

otherwise, f is of type elder butterfly. It should be noted that for the butterfly T. Gaffney [6] found the same condition on Taylor coefficients by studying the structure of $\mathcal{A}$-tangent space (Example 1.4. in [6]). Our approach is more direct by extending the method used in [15,3], and we describe such conditions for all $\mathcal{A}$-types in Rieger's list ($\mathcal{A}$-codimension ≤ 6).

Our second purpose is to demonstrate a systematic use of our criteria for map-germs arising in some specific geometric situation. We develop a method of J.W. Bruce [3] for an application to *extrinsic differential geometry* of surfaces. Look at a generic surface in $\mathbb{R}^3$ from a viewpoint (camera), then we get locally a smooth map from the surface to the plane (screen), that is called the *central projection*. Their singularities have been classified by V.I. Arnold and O.A. Platonova (also O.P. Shcherbak, V.V. Goryunov) [1,2,8,12,22] based on a different framework. It is shown that some germs of $\mathcal{A}$-codimension 5 *do not appear* generically in central projections, although the reason has not been quite clear from the context of $\mathcal{A}$-classification, as Rieger noted in his paper [15]. Our criteria make the reason very clear – the condition of intrinsic derivatives $\eta^k\lambda$ determines jets of Monge form of the surface, while the condition of Taylor coefficients determines a special position of viewpoints (Remarks 4.11 and 4.12). We present an alternative transparent proof of Arnold–Platonova's theorem within the $\mathcal{A}$-classification theory, moreover, we classify singularities arising in central projections of *moving surfaces* with one-parameter in 3-space (Theorem 4.6).

As a byproduct, in another paper [21] we obtain a generalization of projective classification of jets of Monge forms by Platonova [12]. Our criteria are also useful to determine the bifurcation diagrams of map-germs, especially of corank two. See [24,25] for the detail.

The rest of this paper is organized as follows. In §2 we briefly introduce the classification of plane-to-plane map-germs. In §3 we give a complete set of criteria for all $\mathcal{A}$-types with $\mathcal{A}$-cod ≤ 6. In §4 we show an application of our criteria to the central projection of smooth surfaces.