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Chain recurrent sets of generic mappings on compact spaces

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A B S T R A C T

Let 0-CR denote the class of all metric compacta X such that the set of maps $f : X \to X$ with 0-dimensional sets $CR(f)$ of chain recurrent points is a dense G_δ-subset of the mapping space $C(X,X)$ (with the uniform convergence). We prove, among others, that countable products of polyhedra or locally connected curves belong to 0-CR. Compacta that admit, for each $\epsilon > 0$, an ϵ-retraction onto a subspace from 0-CR belong to 0-CR themselves. Perfect ANR-compacta or n-dimensional LC^{n-1}-compacta have perfect $CR(f)$ for a generic self-map f. In the cases of polyhedra, compact Hilbert cube manifolds, local dendrites and their finite products, a generic f has $CR(f)$ being a Cantor set and the set of periodic points of f of arbitrarily large periods is dense in $CR(f)$. The results extend some known facts about $CR(f)$ of generic self-maps f on PL-manifolds.

1. Introduction

For compact metric spaces X, Y, we denote by $C(X,Y)$ the space of all continuous maps $f : X \to Y$ with the topology of uniform convergence. If d is a metric in Y, then $\hat{d}$ denotes the metric of uniform convergence in $C(X,Y)$, $\hat{d}(f,g) = \sup\{d(f(x), g(x)) : x \in X\}$. We say that a map $f \in C(X,X)$ with a property P is *generic* if the set of all maps in $C(X,X)$ with property $\mathcal{P}$ is residual (i.e., contains a dense G_δ-subset) in $C(X,X)$.

In this paper we investigate chain recurrent points of generic maps on some compacta. Recall that, given a map $f : (X,d) \to (X,d)$ and $\epsilon > 0$, a finite set $\{x = x_0, x_1, \ldots, x_n = y\} \subset X$ is an *ϵ-chain* for f from x

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to y if $d(f(x_{i-1}), x_i) < \epsilon$ for each $i = 1, \ldots, n$; a point x is called a *chain recurrent point* of f if, for each $\epsilon > 0$, there exists an ϵ-chain for f from x to x (sometimes, x is said to have a periodic ϵ-pseudo-orbit for each ϵ). The set of chain recurrent points of f is denoted by $CR(f)$. Clearly, it contains the set $Per(f)$ of periodic points of f.

The notion of a chain recurrent point plays an important role in dynamical systems. Some general properties of it can be found, e.g., in books [1,3,6]. There is an extensive literature devoted to generic properties of maps involving chain recurrency and related notions, where they were studied mainly for diffeo- or homeomorphisms on smooth or PL-manifolds (see, e.g., [25,26,13,14,23,2]).

In this paper, by a dimension of a metric separable space we mean its covering (or, equivalently, inductive) dimension (see [12]). Denote by 0-CR the family of all compacta X such that the set $CR(f)$ is 0-dimensional for a generic map $f \in C(X, X)$. It is known that all finite graphs and PL-manifolds are in class 0-CR (see [27] and [2], resp.). Generalizing these results, we prove in Section 3 that all finite polyhedra belong to class 0-CR. In Section 4, we show that all locally connected, 1-dimensional continua are elements of 0-CR, as well. Moreover, using a small-retractions-technique, we derive similar results for all Menger compacta, compact Hilbert cube manifolds and some important non-locally connected continua. The technique also gives us immediately that all countable (finite or infinite) products as well as cones and suspensions over these compacta are members of 0-CR.

A stronger property of a generic map f, of having $CR(f)$ homeomorphic to the Cantor set $\mathcal{C}$, is formulated in [2] for PL-manifolds (the proof in [2], however, is only barely sketched at the very end of the memoir). In Section 6 we provide a short proof that $CR(f)$ has no isolated points for a generic map $f : X \to X$ if X is a perfect ANR or X is a perfect n-dimensional LC^{n-1}-compactum; moreover, if such X has the local periodic point property, then $Per(f)$ has no isolated points and is dense in $CR(f)$; moreover, for each integer $l \geq 2$, periodic points of periods $\geq l$ form a dense subset of $CR(f)$. In particular, if X is a finite polyhedron, a compact Hilbert cube manifold, a local dendrite or a finite product of these spaces, then $CR(f)$ is a Cantor set containing $Per(f)$ as a dense subset for a generic f.

Sometimes we can also claim that a generic self-map is zero-dimensional. This is true in the cases of ANR-compacta with coinciding finite dimensions dim $=$ *ped*, where *ped* is the piecewise embedding dimension in the sense of [16], Menger manifolds and countable products of locally connected curves, among others.

Finally, we observe that a measure-theoretical result of [27] that $CR(f)$ is of measure zero for a generic map $f \in C(X, X)$, if X is an n-dimensional LC^{n-1} perfect compactum equipped with a Borel, finite, non-atomic measure, extends on all mentioned above compacta.

2. Some general properties

The following characterization of chain recurrent points is very useful.

Proposition 2.1. *([7]) For any compact space X and $f \in C(X, X)$, $x \notin CR(f)$ if and only if there exists an open set $U \subset X$ such that $x \notin U$, $f(x) \in U$ and $f(\overline{U}) \subset U$.*

It follows easily from this proposition that (for each compact space X and $f \in C(X, X)$), the set $CR(f)$ is nonempty, closed and invariant under f [6].

We will use the following fact.

Fact 2.2. *([7]) The set-valued function*

$$CR : C(X, X) \to 2^X, \quad f \mapsto CR(f)$$

defined on a compact metric space X, is upper semi-continuous.