Topology and its Applications 202 (2016) 318–324

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

The Baire property in the compact-open topology of Lašnev spaces

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ARTICLE INFO

Article history:
Received 11 July 2015
Received in revised form 1 January 2016
Accepted 25 January 2016

MSC:
54E52
54C35

Keywords:
Compact-open topology
Baire property
Moving off property
Lašnev spaces

ABSTRACT

Finding an internal condition on a topological space X which is necessary and sufficient for the compact-open topology $C_k(X)$ to be Baire is an open problem. The moving off property is a known characterization for the class of first-countable or locally compact spaces. Here we show that it also holds for the class of closed images of first-countable paracompact spaces, hence Lašnev spaces in particular; furthermore, Baireness of $C_k(X)$ is equivalent to its α-favorability for X in this class. We will also show that if X is the closed image of a locally compact paracompact space then $C_k(X)$ is α-favorable.

1. Introduction

A natural question that arises with $C_k(X)$, the space of all continuous real-valued functions on X with the compact-open topology, is: given a topological property P of $C_k(X)$ is there a property Q of X such that $C_k(X)$ has property P if and only if X has property Q? In particular, it is an open problem to characterize those spaces X for which $C_k(X)$ has the Baire property. In [3] it was shown that if X is a q-space (e.g., first-countable or locally compact), then $C_k(X)$ is Baire if and only if X has the moving off property (see the next section for definitions). In [1] and [2] the moving off property is also shown to characterize those X for which $C_k(X)$ is Baire in the class of generalized ordered spaces (GO-spaces), and for certain examples of topological fans with topologies intermediate to the metric and sequential fans. In this paper, we will show that if X is a Lašnev space, the closed image of a metric space, then the moving off property also characterizes when $C_k(X)$ is Baire; more generally, this holds in the class of closed images of first-countable

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http://dx.doi.org/10.1016/j.topol.2016.01.026

paracompact spaces. We also show that for X in this class, $C_k(X)$ is Baire iff it is α-favorable. Finally, we prove that if X is the closed image of a locally compact paracompact space then $C_k(X)$ is α-favorable.

All spaces in this paper are assumed to be completely regular and T_1.

2. Preliminaries

A collection $\mathcal{A}$ of sets is said to *move off* a collection $\mathcal{B}$ of sets if for any $B \in \mathcal{B}$ there is an $A \in \mathcal{A}$ such that $A \cap B = \emptyset$. Recall that a collection $\mathcal{S}$ of subsets of a topological space is *discrete* if given any $x \in X$ there is an open set U containing x such that $U \cap A \neq \emptyset$ for at most one $A \in \mathcal{S}$. A collection $\mathcal{S}$ of subsets of a topological space is *strongly discrete* if there is there is a discrete family $\mathcal{U} = \{U_A : A \in \mathcal{S}\}$ of open sets such that $A \subseteq U_A$ for all $A \in \mathcal{S}$. Gruenhage and Ma [3] defined the following property. A topological space X is said to have the *moving off property* if given any collection $\mathcal{K}$ of compact subsets of X that moves off the family of all compact subsets, there exists a strongly discrete infinite $\mathcal{K}' \subseteq \mathcal{K}$. In [3] it was shown that for any X if $C_k(X)$ is Baire then X has the moving off property, and the converse was conjectured.

Conjecture 1. *X has the moving off property if and only if $C_k(X)$ is Baire.*

The analogous characterization for $C_p(X)$, in which (roughly speaking) compact sets in the definition of the moving off property are replaced by finite sets, is true and was given independently by E.G. Pytkeev [7], V.V. Tkachuk [8,9], and E.K. van Douwen (unpublished; see page 34 in [10]).

Conjecture 1 has been shown to be true when restricted to various classes of topological spaces. The following is a well known generalization of first countability. A space X is said to be a *q-space* if for each $x \in X$ there is a sequence $(U_i : i \in \omega)$ of open neighborhoods of x such that if $x_n \in U_n$ for each $n \in \omega$ then $\{x_n : n \in \omega\}$ has a cluster point.

Theorem 2.1. *([3])*

(a) *If X is a q-space with the moving off property, then X is locally compact.*
(b) *If X is locally compact, then $C_k(X)$ is Baire if and only if X has the moving off property.*

We now recall game characterizations for both the moving off property and the Baire property which will be used extensively in showing that Conjecture 1 holds in the class of Lašnev spaces.

Recall that the Banach–Mazur Game, $BM(X)$, on a topological space X is a game with two players β and α, which alternately play non-empty open subsets of X. On move 0, β chooses a non-empty open set U_0 and α responds with a non-empty open set $V_0 \subseteq U_0$. On move $n > 0$, β chooses a non-empty open set $U_n \subseteq V_{n-1}$ and α responds with a non-empty open set $V_n \subseteq U_n$. β wins if $\bigcap\{U_i : i \in \omega\} = \emptyset$; α wins otherwise. A strategy for a player is *stationary* if it depends only on the last move of the opponent. The space X said to be *weakly α-favorable* (resp., *α-favorable*) if α has a winning strategy (resp., stationary winning strategy) in $BM(X)$. The following is due to Oxtoby:

Theorem 2.2. *([6]) β has no winning strategy in $BM(X)$ if and only if X is Baire.*

The following game is found in [3]. Suppose X is a topological space; the game $G^{\circ}_{K,L}(X)$ is a topological game with two players K and L, which alternately choose compact subsets of X. On move 0, K plays a compact set K_0 and L responds with a compact set L_0 such that $K_0 \cap L_0 = \emptyset$. On move $n > 0$, K chooses a compact set K_n with no restrictions, and L responds with a compact set L_n such that $L_n \cap \bigcup\{K_i : i \leq n\} = \emptyset$. K wins if $\{L_i : i \in \omega\}$ is strongly discrete. L wins otherwise.