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On blockers in continua

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ABSTRACT

We continue in the study of blockers in continua that were first defined by Illanes and Krupski. Especially, we are dealing with the following question of these authors. For a given continuum, if each closed set that blocks any finite set also blocks any closed set, does it imply that the continuum is locally connected? We provide a negative answer by constructing a planar non-locally connected lambda-dendroid in which every closed set which blocks every finite set also blocks every closed set. On the other hand we prove that in the realm of hereditarily decomposable chainable continua or among smooth dendroids the answer is positive. Finally we compare the notion of a non-blocker with the notion of a shore set and we show that the union of finitely many mutually disjoint closed shore sets in a smooth dendroid is a shore set. This answers a question of the authors.

1. Introduction

In this paper, all the unexplained notions can be found in the book [10]. We only recall that for a continuum X, we consider the following hyperspaces equipped with the Hausdorff metric and the corresponding topology:

$$2^X = \{A \subseteq X :\ A \text{ is closed and nonempty}\}$$
$$F_n(X) = \{A \in 2^X :\ A \text{ has at most } n \text{ points}\}$$
$$F(X) = \bigcup_n F_n(X).$$

For a pair of points a, b in a uniquely arcwise connected continuum X we denote by ab the unique arc joining the points a and b. In what follows we mainly study the notion of a blocker which was introduced

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in [6]. The original definition is using arcs in hyperspaces, but the one used here is known to be equivalent as proved in [6, Proposition 1.3].

Definition 1.1. For $A, B \in 2^X$ we say that B does not *block* A if $A \cap B = \emptyset$ and the union of all subcontinua of X intersecting A and contained in $X \setminus B$ is dense in X. When B blocks A, we say that B is a blocker of A. For a subset $\mathcal{H} \subseteq 2^X$ we use the notation

$$\mathcal{B}(\mathcal{H}) = \{B \in 2^X \colon\ B \text{ blocks each element of } \mathcal{H}\},$$

i.e., the set $\mathcal{B}(\mathcal{H})$ consists of all blockers of $\mathcal{H}$.

As a reasonable counterpart to this the set $\mathcal{NB}(\mathcal{H})$ is defined as follows.

$$\mathcal{NB}(\mathcal{H}) = \{N \in 2^X \setminus \{X\} \colon N \text{ does not block each element of } \mathcal{H}, \text{ which is disjoint with } N\}.$$

The notion $\mathcal{NB}(F_1(X))$ was first used in [5] to characterize the simple closed curve, trees and dendrites. Since the notion of a non-blocker was used rather informally in [5], we are allowed to settle the following definition.

Definition 1.2. Let X be a continuum and let $A \subseteq X$. We say that A is a *non-blocker* if there is an increasing sequence (K_n) of continua disjoint with A whose union is dense in X.

Our definition is equivalent to the conditions stated in [5, Proposition 2.2]: One can easily check that $N \in 2^X$ is a non-blocker if and only if $N \in \mathcal{NB}(\{S\})$ for some $S \in F_1(X)$. Thus definition of a non-blocker is related with the notion $\mathcal{NB}$ naturally.

The notion of a non-blocker resembles the notion of a shore set as defined first in [9]. These are not equivalent in the general case, but as shown in the last chapter, there are some classes of dendroids in which the notions coincide. The notion of a non-blocker generalizes straightforwardly the notion of a non-block point defined in [3] in which one of the main results states that every nontrivial continuum is spanned by the set of all non-block points [3, Theorem 2.8].

In [6, Theorem 1.5] the authors proved that for every locally connected continuum X the equality

$$\mathcal{B}(F(X)) = \mathcal{B}(2^X) \tag{1}$$

holds and they asked the following question [6, Question 1.6]:

Question 1.3. Does the equality (1) characterize the local connectedness of X?

In spite of that we show that the general answer to this question is negative (Example 3.4) we prove that in some classes of spaces Question 1.3 has a positive answer. Namely we show this for the class of hereditarily decomposable chainable continua and for the class of smooth dendroids (Proposition 2.1, Proposition 2.3). In the last section we answer positively one of the questions from [2, Table 1] by the use of the notion of non-blockers.

Let us now recall some facts about smooth dendroids. First of all a dendroid X is said to be *smooth* if there is a point $p \in X$ such that for every sequence (x_n) converging to some x the sequence (px_n) converges to px. Let X be a dendroid. A metric ρ on X is called *radially convex* with respect to p if $\rho(p, x) < \rho(p, y)$ if $x \in py$ and $x \neq y$. The characterization of smooth dendroids by radially convex metrics was proved in [4, Theorem 10].