Topology and its Applications 202 (2016) 356–365

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

A note on spaces that are finitely an F-space ☆

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A R T I C L E I N F O

Article history:
Received 28 October 2015
Received in revised form 18 January 2016
Accepted 29 January 2016

MSC:
54E17
13A15
18A40
06D22

Keywords:
F-space
Finitely an F-space
Frame
F-frame
Finitely an F-frame

A B S T R A C T

A Tychonoff space X is finitely an F-space if βX is expressible as a union of finitely many closed F-spaces. Larson [13] has shown that, for normal spaces X, the property of being finitely an F-space can be characterized in terms of algebraic properties of the ring $C(X)$. By extending this notion to locales, we show that the normality restriction can actually be dropped, even in spaces, and thus sharpen Larson's result.

1. Introduction

All spaces in this note are Tychonoff, which is to say they are completely regular and Hausdorff. There are various topological properties of a space X which can be characterized in terms of algebraic properties of the ring $C(X)$ of continuous real-valued functions on X. For instance, X is a P-space (meaning that every G_δ-set is open) precisely when $C(X)$ is von Neumann regular, and X is an F-space (meaning that every cozero-set is C^*-embedded) if and only if every finitely generated ideal in $C(X)$ is principal. See [9] for other such properties.

A space X is *finitely an F-space* if βX can be written as a union $\beta X = K_1 \cup \cdots \cup K_n$, where each K_i is a closed set in βX and is an F-space in the subspace topology. These spaces were first considered in [10],

☆ The authors acknowledge financial assistance from the National Research Foundation of South Africa via a grant held by the first-named author with Grant Number 93514.

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http://dx.doi.org/10.1016/j.topol.2016.01.030

and have since been studied by Larson in a series of papers, including [13] and [14]. In the former paper she gives a characterization, among normal spaces, in terms of an algebraic condition on the ring $C(X)$. In fact, she shows that the condition is sufficient for X to be finitely an F-space with no normality assumed, and necessary if X is normal. Thus, for normal spaces there is an algebraic characterization.

Our goal in this note is to unshackle the characterization from normality. This we achieve by working with frames instead of spaces. We thus have to define, in a conservative way, what it means to say a frame is finitely an F-frame. By "conservative" we mean that a space must be finitely an F-space if and only if the frame of its open sets is finitely an F-frame.

We shall then exploit the presence of the Lindelöf coreflection, λL, of a frame L, the normality of the frame λL, and the fact that the rings $\mathcal{R}L$ and $\mathcal{R}(\lambda L)$ are isomorphic. Loosely speaking, one can say the normality that one needs to assume in spaces is already present (albeit at a higher level) if we work with frames.

To some extent our pattern of proofs will be modeled on that of Larson. Indeed, in one instance (the proof of the right-to-left implication of Theorem 3.9) we will piggyback on her proof of the corresponding implication in spaces.

2. Preliminaries

2.1. Frames and their homomorphisms

Our general references for frames are [11] and [17], and our notation is standard. As usual, we denote by βL the Stone–Čech compactification of a completely regular frame L, and we write $j_L\colon \beta L \to L$ for the coreflection map from compact completely regular frames to L. The right adjoint of j_L is denoted by r_L. Recall that, for any $a \in L$, $r_L(a) = \{x \in L \mid x \prec\!\!\prec a\}$. We write $\operatorname{Coz} L$ for the set of cozero elements of L. For any $c, d \in \operatorname{Coz} L$, $r_L(c \vee d) = r_L(c) \vee r_L(d)$. If L is normal, then $r_L(a \vee b) = r_L(a) \vee r_L(b)$ for all $a, b \in L$. A frame homomorphism $h\colon L \to M$ is *coz-onto* if for every $d \in \operatorname{Coz} M$, there is a $c \in \operatorname{Coz} L$ such that $h(c) = d$.

We denote by λL the Lindelöf coreflection of L. See [15] for details. By a *quotient map* we mean a surjective frame homomorphism. For a frame L and $a \in L$, we write $\kappa_a\colon L \to {\uparrow}a$ for the quotient map $\kappa_a(x) = a \vee x$. Recall that if $h\colon L \to M$ is a quotient map, we have the dense factorization

$$L \xrightarrow{\kappa_{h_*0}} {\uparrow}h_*(0) \xrightarrow{\bar{h}} M$$

where $\bar{h}\colon {\uparrow}h_*(0) \to M$ maps as h, and is a dense quotient map representing the "closure of M in L". We shall at times suppress the name of the mapping, and simply write ${\uparrow}h_*(0) \to M$.

In the proof of the main result we shall assume the necessary foundations that make every compact regular frame spatial.

2.2. C^*-quotients

Our approach to the ring $\mathcal{R}L$ follows that of [3], so that $\mathcal{R}L$ is the f-ring whose members are the frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to L$, where $\mathfrak{L}(\mathbb{R})$ is the frame of reals. Recall from [1] that a quotient map $h\colon L \to M$ is called a *C^*-quotient map*, and we then say M is a *C^*-quotient* of L, in case for every bounded $\alpha \in \mathcal{R}M$ there exists some $\bar{\alpha} \in \mathcal{R}L$ such that the triangle below commutes.