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Invariant measures on topological groups ☆

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ABSTRACT

In this paper, we prove that every locally $\mathcal{K}$ (see Definition 3.4) topological group has a nonzero outer regular invariant Borel measure when $\mathcal{K}$ is an admissible invariant family which is separated by $\mathcal{N}_G$. In this case, every open set and every member of $\mathcal{S}(\mathcal{K}_0)$ are $\mathcal{K}$-inner regular. This extends the existence theorem of Haar measure on locally compact Hausdorff groups.

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1. Introduction

A Haar measure is a nonzero Borel measure μ on a locally compact Hausdorff topological group G, such that

(1) $\mu(gE) = \mu(E)$ for any Borel set E and any $g \in G$;
(2) $\mu(K) < +\infty$ for any compact set;
(3) μ is outer regular and every open set is inner regular.

The concept of the Haar measure was introduced by Alfred Haar in 1933 [1]. He proved the existence of an invariant measure for locally compact topological groups with a countable basis. In this case, the invariant measure is called a Haar measure.

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The existence and uniqueness of a Haar measure for locally compact Hausdorff topological groups were first proven in general by Andre Weil in 1940 [2]. In fact, Weil constructed a Haar integral on a given locally compact Hausdorff group G. By Riesz representation theorem, there exists a Haar measure for G.

In the late 1940s, based on the method used by Haar and Weil, Paul R. Halmos gave a proof about the existence of Haar measure for locally compact groups in terms of “content” [3]. He defined an invariant content λ on the family of all compact subsets, used the term “Borel set” for elements of the σ-ring generated by compact sets, and then constructed a Haar measure from λ.

The main purpose of this paper is to extend the existence theorem of a Haar measure on locally compact Hausdorff groups to some locally $\mathcal{K}$ (see Definition 3.4) topological groups. First, we introduce the notion of admissible families on a topological space, and give some properties of a content λ defined on an admissible family $\mathcal{K}$. Afterwards, we discuss the regularity of the measure induced by λ (see Theorem 2.2). Based on Theorem 2.2, we construct a nonzero outer regular invariant Borel measure μ for some locally $\mathcal{K}$ topological groups, and show that every open set and every member of $\mathcal{S}(\mathcal{K}_0)$ are $\mathcal{K}$-inner regular (see Theorem 3.1). Actually, Theorem 3.1 is a generalization of the well-known existence theorem of Haar measure on a locally compact Hausdorff group (see Corollary 3.1).

Throughout this paper, undefined terminology should be referred to [5].

2. Contents and $\mathcal{K}$-inner regularity

In this section, we introduce the notion of admissible families, and discuss the regularity of the measure induced by λ which is a content defined on an admissible family $\mathcal{K}$.

Definition 2.1. Let $(X, \mathcal{B}, \mu)$ be a measure space and $\mathcal{F} \subset \mathcal{B}$. We say that a measurable set E is *$\mathcal{F}$-inner regular* if $\mu(E) = \sup\{\mu(A) : A \subset E, A \in \mathcal{F}\}$. If every $E \in \mathcal{B}$ is $\mathcal{F}$-inner regular, then μ is said to be *$\mathcal{F}$-inner regular.*

Remark. This definition is accordance with that of $\mathcal{K}$-inner regular in [4].

Proposition 2.1. *Let $(X, \mathcal{B}, \mu)$ be a measure space and $\mathcal{F} \subset \mathcal{B}$. Then every member of $\mathcal{F}$ is $\mathcal{F}$-inner regular.*

Proposition 2.2. *Let $(X, \mathcal{B}, \mu)$ be a measure space and $\mathcal{F}$ a family of measurable sets which is closed under the formation of countable intersections and finite unions. If for each $n \in \mathbb{N}^*$, E_n is a finite measure set which is $\mathcal{F}$-inner regular, then $\bigcap_{n=1}^{\infty} E_n$ and $\bigcup_{n=1}^{\infty} E_n$ both are $\mathcal{F}$-inner regular.*

The proofs of Proposition 2.1 and Proposition 2.2 are trivial.

Throughout this section, unless in a special context we explicitly say otherwise, we assume that $(X, \mathcal{T}, \mathcal{B}, \mu)$ is a topological measure space where $(X, \mathcal{B}, \mu)$ is a measure space and $\mathcal{T}$ is a topology on X such that $\mathcal{T} \subset \mathcal{B}$, that is, every Borel set is measurable. Let $\mathcal{F}$ be a family of measurable subsets of X which is closed under the formation of countable intersections and finite unions. We shall use the symbol $\mathcal{F}_0$ for the family of all finite measure sets which belong to $\mathcal{F}$.

Proposition 2.3. *Let A be a member of $\mathcal{F}$. If there exists an open set U with finite measure such that $A \subset U$, and every open set is $\mathcal{F}$-inner regular, then every open subset of A is $\mathcal{F}$-inner regular.*

Proof. Let E be an open subset of A. Suppose $E = V \cap A$ and V is an open subset of X. Since $V \cap U$ and A both are $\mathcal{F}$-inner regular, by Proposition 2.2, we have that $E = V \cap U \cap A$ is $\mathcal{F}$-inner regular. □

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