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On the archimedean kernels of function rings in pointfree topology

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ABSTRACT

The article explores for which function rings A on a frame L all the archimedean kernels of A are determined by the frame homomorphisms from L. It turns out that, for suitable A, this property is equivalent to three different conditions concerning the relation between A and $\mathfrak{R}L$, the ℓ-ring of all continuous real-valued functions on L.

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Recall that an archimedean kernel of an ℓ-ring A is an ℓ-ring ideal J of A for which A/J is archimedean, or, expressed in elementary terms, such that

$$\text{for any } a, b \geq 0 \text{ in } A, \text{ if } (na - b)^+ \in J \text{ for all } n = 1, 2, \ldots \text{ then } a \in J.$$

Now, if A is a sub-ℓ-ring of $\mathfrak{R}L$, the ℓ-ring of all continuous real-valued functions on some frame L, always understood to contain the unit $\mathbf{1}$ of $\mathfrak{R}L$, the most obvious archimedean kernels of A are those arising from L, that is, the

$$J = A \cap \mathrm{Ker}(\mathfrak{R}h) = \{\gamma \in A \mid \mathfrak{R}h(\gamma) = \mathbf{0}\} = \{\gamma \in A \mid h\gamma = \mathbf{0}\}$$

for some frame homomorphism $h : L \to M$, where $\mathfrak{R}h : \mathfrak{R}L \to \mathfrak{R}M$, $\gamma \mapsto h\gamma$, is the ℓ-ring homomorphism determined by h. Calling the sub-ℓ-rings A of some $\mathfrak{R}L$ the *function rings on* L and the $J = \{\gamma \in A \mid \mathfrak{R}h(\gamma) = \mathbf{0}\}$ the *L-based* archimedean kernels of A, a natural condition concerning a function ring A on a frame L is that *all archimedean kernels of A are L-based*. The main purpose of this note is to show this and three other conditions on A are equivalent.

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We begin with a brief account of the concepts and facts to be used here. As general references, we suggest Banaschewski [3] and [4] concerning real-valued continuous functions on frames, and Picado and Pultr [8] for frames in general. For the background of [4] see Ball–Hager [1] and Madden [7].

The basic setting here is given by the adjoint functors

$$\mathfrak{R} : \mathbf{CRFrm} \to \mathbf{A} \text{ and } \mathfrak{K} : \mathbf{A} \to \mathbf{CRFrm}$$

between the category **CRFrm** of completely regular frames and the category **A** of archimedean f-rings with unit 1, where $\mathfrak{R}L$, as mentioned already, is the ℓ-ring of all real-valued continuous functions on L, that is, the frame homomorphisms from the frame $\mathfrak{L}(\mathbf{R})$ of reals to L, with the usual ℓ-ring operations derived from those of $\mathbf{Q}$, and $\mathfrak{K}A$ is the frame of all archimedean kernels of the archimedean f-ring A with unit, with the adjunction maps

$$\lambda_A : A \to \mathfrak{R}\mathfrak{K}A,\ a \mapsto \hat{a}, \quad \hat{a}(p,q) = \langle (na-k)^+ \wedge (\ell - na)^+ \rangle,$$

$k, \ell \in \mathbf{Z}$ and $n = 1, 2, \dots$ such that $p = \frac{k}{n}$ and $q = \frac{\ell}{n}$, where $\langle \cdot \rangle$ indicates the principal archimedean kernel generated by $\cdot$, and

$$\mu_L : \mathfrak{K}\mathfrak{R}L \to L,\ \mu_L(\langle \gamma \rangle) = \operatorname{coz}(\gamma) = \bigvee \{\gamma(-n,0) \vee \gamma(0,n) \mid n = 1, 2, \dots\}.$$

Note that the definition of the above $\hat{a}$ simplifies to

$$\langle (a-p)^+ \wedge (q-a)^+ \rangle$$

whenever A is an algebra over $\mathbf{Q}$, and p and q stand for $p1$ and $q1$ for the unit $1 \in A$.

Now, for any function ring A on L, we have the homomorphism

$$\mu_L^A : \mathfrak{K}A \xrightarrow{\mathfrak{K}i_A} \mathfrak{K}\mathfrak{R}L \xrightarrow{\mu_L} L,\ \langle \gamma \rangle \mapsto \langle\langle \gamma \rangle\rangle \mapsto \operatorname{coz}(\gamma),$$

where $\langle\langle \cdot \rangle\rangle$ indicates the archimedean kernel generated by $\cdot$ in $\mathfrak{R}L$.

Next, a function ring A on a frame L is said to *separate* L if $\{\operatorname{coz}(\gamma) \mid \gamma \in A\}$ generates L which clearly holds iff μ_L^A is *onto*. Concerning this terminology, note that, for the frame of open sets of a Tychonoff space X, this condition means that the members of A separate points from closed sets in X (Banaschewski–Sioen [6]).

Further, for any A and L of this kind, the covers

$$\{\gamma(p,q) \mid 0 < q-p < \tfrac{1}{n}\}, \quad \gamma \in A,\ p \text{ and } q \text{ in } \mathbf{Q},\ \text{and } n = 1, 2, \dots$$

of L generate a uniformity, the *A-uniformity of* L, and if L is complete with respect to this it is called *A-complete*. In particular, for $A = \mathfrak{R}L$, one calls the $\mathfrak{R}L$-uniformity of L its *real uniformity* and L *realcomplete* iff it is completely regular and $\mathfrak{R}L$-complete.

On the other hand, for any archimedean f-ring A with unit, we have the uniformity on $\mathfrak{K}A$, generated by the covers

$$\{\hat{a}(p,q) \mid 0 < q-p < \tfrac{1}{n}\}, \quad a \in A \text{ and } n = 1, 2, \dots,$$

called the *A-uniformity of* $\mathfrak{K}A$, and as a specific feature concerning the functor $\mathfrak{K}$ we note that $\mathfrak{K}A$ is *complete* with respect to this, as a natural consequence of its adjointness to $\mathfrak{R}$. Indeed, if $h : L \to \mathfrak{K}A$ is the corresponding completion then any $\hat{a} : \mathfrak{L}(\mathbf{R}) \to \mathfrak{K}A$, $a \in A$, is trivially uniform relative to the standard uniformity of $\mathfrak{L}(\mathbf{R})$, and by the completeness of the latter this determines $\bar{a} : \mathfrak{L}(\mathbf{R}) \to L$ such that $h\bar{a} = \hat{a}$.

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