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A compactness result in approach theory with an application to the continuity approach structure

Ben Berckmoes [1]

A R T I C L E I N F O

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Dedicated to Eva Colebunders on the occasion of her 65th birthday

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A B S T R A C T

We establish a compactness result in approach theory which we apply to obtain a generalization of Prokhorov's Theorem for the continuity approach structure.

1. Introduction

Measures of non-compactness [3] have been studied extensively in the context of approach theory [10], on an abstract level [1,2] and in specific approach settings in e.g. hyperspace theory [12], functional analysis [11], function spaces [9] and probability theory [4]. The presence of a vast literature on the interplay between compactness and approach theory is explained by the fact that the latter is a canonical setting which allows for a unified treatment of the classical concept of measure of non-compactness [8].

In this paper we contribute to the knowledge on the interplay between compactness and approach theory. In Section 2 we provide a new compactness result for a general approach space. In Section 3 we apply this result to the specific setting of the so-called continuity approach structure [5,10] to obtain a quantitative generalization of Prokhorov's Theorem.

E-mail address: ben.berckmoes@uantwerpen.be.

[1] The author is postdoctoral fellow at the Fund for Scientific Research of Flanders (FWO).

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2. A compactness result in approach theory

Let X be an approach space with approach system $\mathcal{A} = (\mathcal{A}_x)_{x\in X}$. We first recall some notions related to compactness in X. For more details the reader is referred to [10].

We say that X is *locally countably generated* iff there exists a basis $(\mathcal{B}_x)_{x\in X}$ for $\mathcal{A}$ such that each $\mathcal{B}_x$ is countable.

For $x \in X$, $\phi \in \mathcal{A}_x$ and $\epsilon > 0$ we define the *ϕ-ball with center x and radius ϵ* as the set $B_\phi(x, \epsilon) = \{y \in X \mid \phi(y) < \epsilon\}$. More loosely, we also refer to the latter set as a ball with center x or a ball with radius ϵ.

Consider a point $x \in X$, a sequence $(x_n)_n$ in X and $\epsilon > 0$. We say that $(x_n)_n$ is *ϵ-convergent* to x iff each ball B with center x and radius ϵ contains x_n for all n larger than a certain n_B. We write $x_n \overset{\epsilon}{\to} x$ to indicate that $(x_n)_n$ is ϵ-convergent to x. We define the *limit operator* of $(x_n)_n$ at x as

$$\lambda(x_n \to x) = \inf\left\{\alpha > 0 \mid x_n \overset{\alpha}{\to} x\right\}.$$

We call X *sequentially complete* iff it holds for each sequence $(x_n)_n$ in X that $\inf_{x\in X} \lambda_{\mathcal{A}}(x_n \to x) = 0$ implies the existence of a point x_0 to which $(x_n)_n$ converges (in the topological coreflection).

Let $A \subset X$ be a set. We say that A is *ϵ-relatively sequentially compact* iff every sequence in A contains a subsequence which is ϵ-convergent and we define the *relative sequential compactness index* of A as

$$\chi_{rsc}(A) = \inf\{\alpha > 0 \mid A \text{ is } \alpha\text{-relatively sequentially compact}\}.$$

Notice that relatively sequentially compact sets (in the topological coreflection) have relative sequential compactness index zero, but that the converse does not necessarily hold.

If $(\Phi = (\phi_x)_x) \in \Pi_{x\in X}\mathcal{A}_x$, then a set $B \subset X$ is called a *Φ-ball* iff there exist $x \in X$ and $\alpha > 0$ such that $B = B_{\phi_x}(x, \alpha)$. We call A *ϵ-relatively compact* iff it holds for each $\Phi \in \Pi_{x\in X}\mathcal{A}_x$ that A can be covered with finitely many Φ-balls with radius ϵ and we define the *relative compactness index* of A as

$$\chi_{rc}(A) = \inf\{\alpha > 0 \mid A \text{ is } \alpha\text{-relatively compact}\}.$$

We say that X is *ϵ-Lindelöf* iff it holds for each $\Phi \in \Pi_{x\in X}\mathcal{A}_x$ that X can be covered with countably many Φ-balls with radius ϵ and we define the *Lindelöf index* of X as

$$\chi_L(X) = \inf\{\alpha > 0 \mid X \text{ is } \alpha\text{-Lindelöf}\}.$$

Theorem 2.2, the main result of this section, interconnects the above notions. For its proof we use the following well-known lemma which belongs to the heart of approach theory [10].

Lemma 2.1 *(Lowen). Let $\mathcal{D}_{\mathcal{A}}$ be the set of quasi-metrics d on X with the property that $d(x,\cdot) \in \mathcal{A}_x$ for each $x \in X$. Then the assignment of collections*

$$\mathcal{B}_{\mathcal{D}_{\mathcal{A}},x} = \{d(x,\cdot) \mid d \in \mathcal{D}_{\mathcal{A}}\}$$

is a basis for $\mathcal{A}$.

Theorem 2.2. *Let X be locally countably generated. Then, for any set $A \subset X$,*

$$\chi_{rsc}(A) \le \chi_{rc}(A) \le \chi_{rsc}(A) + \chi_L(X).$$