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Topological groups with many small subgroups

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Dedicated to professor Eva Colebunders for her prominent contributions to categorical topology and closure operators

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A B S T R A C T

We introduce and study a functorial topology on every group G having as a base the family of all subgroups of G. Making use of this topology, we obtain an equivalent description of the small subgroup generating property introduced by Gould [26]; see also Comfort and Gould [6]. This property implies minimal almost periodicity. Answering questions of Comfort and Gould [6], we show that every abelian group of infinite divisible rank admits a group topology having the small subgroup generating property. For unbounded abelian groups of finite divisible rank, we find a new necessary condition for the existence of a group topology having the small subgroup generating property, and we conjecture that this condition is also sufficient.

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All topological groups considered in this paper are assumed to be Hausdorff.

As usual, $\mathbb{Z}$ denotes the group of integers, $\mathbb{Z}(n)$ denotes the cyclic group of order n, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}^+$ denotes the set of all positive natural numbers, $\mathbb{P}$ denotes the set of all prime numbers and $\mathfrak{c}$ denotes the cardinality of the continuum. We write $G \cong H$ when groups G and H are isomorphic.

Let G be a group. For $m \in \mathbb{N}$, we let

$$mG = \{mg : g \in G\} \quad \text{and} \quad G[m] = \{g \in G : mg = e_G\},$$

where e_G is the identity element of G. (When G is abelian, we use 0 instead of e_G.) A group G is *torsion* if $G = \bigcup_{m\in\mathbb{N}^+} G[m]$. A group G has *finite exponent* if $mG = \{e_G\}$ for some integer $m \geq 1$. The smallest integer m with this property is called the *exponent* (or the *order*) of G. A group of finite exponent is said to be a *bounded torsion* (or simply *bounded*) group. For a cardinal σ, we use $G^{(\sigma)}$ to denote the direct sum of σ many copies of the group G. For a subset X of G we use $\langle X\rangle$ to denote the smallest subgroup of G containing X. For $x \in G$ we write $\langle x\rangle$ instead of $\langle\{x\}\rangle$.

For a subset X of a topological group G, we use $\overline{X}$ to denote the closure of X in G.

Our terminology and notation follow [10,22].

1. The small subgroup generating property

A subset X of a topological group G *topologically generates* G provided that $\langle X\rangle$ is dense in G, i.e., $\overline{\langle X\rangle} = G$.

The following definition was given in [26]; see also [6,27].

Definition 1.1. A topological group G has the *small subgroup generating property* (shortly, SSGP) if for every neighborhood U of the identity e_G, there exists a family $\mathcal{H}$ of subgroups of G such that $\bigcup\mathcal{H} \subseteq U$ and $\bigcup\mathcal{H}$ topologically generates G.

Furthermore, an infinite series of properties SSGP(n) was defined in [6, Definition 3.3].

Definition 1.2. Let G be a topological group.

(a) G has SSGP(0) if G is the trivial group;
(b) for an integer $n > 0$, G has SSGP(n) provided that, for every neighborhood U of e_G, there exists a family $\mathcal{H}$ of subgroups of G such that $\bigcup\mathcal{H} \subseteq U$, the closed subgroup N of G topologically generated by $\bigcup\mathcal{H}$ is normal and G/N has SSGP($n-1$).

Clearly, the classes SSGP and SSGP(1) coincide.

These notions are important because they are ultimately related to the classical notion of a minimally almost periodic group; see (1) below.

According to von Neumann's terminology [32], a topological group G is called:

(a) *minimally almost periodic* if every continuous homomorphism $G \to K$ to a compact group K is trivial;
(b) *maximally almost periodic* if continuous homomorphisms $G \to K$ into compact groups K separate the points of G.

It was proved in [6, Remark 3.4, Theorem 3.5] that

$$\text{SSGP} = \text{SSGP}(1) \to \text{SSGP}(2) \to \ldots \to \text{SSGP}(n) \to \ldots \to \text{ minimally almost periodic.} \tag{1}$$