Topology and its Applications 200 (2016) 133–145

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Completeness number of families of subsets of convergence spaces

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A R T I C L E   I N F O

Article history:
Received 30 January 2015
Received in revised form 21 May 2015
Available online 4 January 2016

In honor of Professor Eva Colebunders

Keywords:
Convergence theory
Completeness
Compactness

A B S T R A C T

Compactoid and compact families generalize both convergent filters and compact sets. This concept turned out to be useful in various quests, like Scott topologies, triquotient maps and extensions of the Choquet active boundary theorem.
The completeness number of a family in a convergence space is the least cardinality of collections of covers for which the family becomes complete. 0-completeness amounts to compactness, finite completeness to relative local compactness and countable completeness to Čech completeness. Countably conditional countable completeness amounts to pseudocompleteness of Oxtoby. Conversely, each completeness class of families can be represented as a class of conditionally compactoid families. In this framework, the theorem of Tikhonov for compactoid filters becomes a special case of the theorem on the completeness number of products.
A characterization of completeness in terms of non-adherent filters not only provides a unified language for convergence and completeness, but also clarifies preservation mechanisms of completeness number under various operations.

1. Introduction

Completeness of a convergence is a notion relative to that of *fundamental* (or *Cauchy*) filter. An abstract approach to completeness consists in declaring a class $\mathbb{C}$ of filters to be *Cauchy* filters whenever if $\mathcal{F} \in \mathbb{C}$ and $\mathcal{F} \subset \mathcal{G}$ then $\mathcal{G} \in \mathbb{C}$, if $\mathcal{F}, \mathcal{G} \in \mathbb{C}$ then $\mathcal{F} \cap \mathcal{G} \in \mathbb{C}$ and if all principal ultrafilters belong to $\mathbb{C}$, was adopted in the book [14] of Eva Colebunders.

Two types of completeness of topological spaces have been mainly considered in the literature. The first qualifies a topology as *complete* if each fundamental filter is *convergent* (for example, metric completeness), the second, if each fundamental filter is *adherent* (for instance, Čech completeness). In some cases the two notions coincide; this happens, for example, in metric spaces where fundamental filters are defined as those that contain elements of arbitrarily small diameter.

In this paper, I shall study the second type of completeness in a broader framework of convergence spaces. Traditionally, fundamental filters have been defined in terms of collections of covers. In [4] I proposed to

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http://dx.doi.org/10.1016/j.topol.2015.12.016
0166-8641/

relate fundamental filters to collections of non-adherent families, a *non-adherent family* being a dual concept of that of *cover*. An advantage of this dual approach is to make evident the reasons of preservation of completeness under several types operations.

A *preconvergence* ξ on a set X is a relation between $\mathbb{F}(X)$ (the set of filters on X) and X, denoted by $x \in \lim_\xi \mathcal{F}$, such that $\mathcal{F} \subset \mathcal{G}$ implies that $\lim_\xi \mathcal{F} \subset \lim_\xi \mathcal{G}$. A preconvergence ξ is a *convergence* if $x \in \lim_\xi \{x\}^\uparrow$, where $\{x\}^\uparrow := \{A \subset X : x \in A\}$ is the principal ultrafilter of x. Then we say that $\mathcal{F}$ *converges* to x, equivalently, x is a *limit* of $\mathcal{F}$.[1,2] We denote by $|\xi|$ the underlying set of ξ.

We say that A *meshes* B

$$A\#B$$

if $A \cap B \neq \varnothing$. This simple but useful relation is of course symmetric. It extends to families of sets: $\mathcal{A}\#\mathcal{B}$ means that $A\#B$ for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$. If $\mathcal{A} = \{A\}$ then we write $A\#\mathcal{B}$ to abridge $\{A\}\#\mathcal{B}$. The *grill* $\mathcal{A}^\#$ of a family of subsets of a set X was defined by G. Choquet in [1] as

$$\mathcal{A}^\# := \{H \subset X : H\#\mathcal{A}\}.$$

The *adherence* of a family $\mathcal{H}$ of subsets of a convergence space is defined by

$$\mathrm{adh}_\xi \mathcal{H} := \bigcup_{\mathcal{F}\#\mathcal{H}} \lim_\xi \mathcal{F}.$$

We say that a family $\mathcal{H}$ on $|\xi|$ is *adherent* if $\mathrm{adh}_\xi \mathcal{H} \neq \varnothing$, and *non-adherent* if $\mathrm{adh}_\xi \mathcal{H} = \varnothing$.

A family $\mathcal{P}$ of subsets of a convergence space X is said to be a *cover* of a subset A of X

$$\mathcal{P} \succ_\xi A \tag{1.1}$$

provided that $\mathcal{F} \cap \mathcal{P} \neq \varnothing$[3] for each filter $\mathcal{F}$ such that $A\# \lim_\xi \mathcal{F}$.[4] It is a simple but fundamental fact [3] that

$$\mathcal{P} \succ_\xi A \Longleftrightarrow \mathrm{adh}_\xi \mathcal{P}_c \cap A = \varnothing. \tag{1.2}$$

2. Compactoid families

Compact families constitute a common generalization of compact sets and convergent filters. The concept of compact filters was studied in [12,7] and an akin notion was applied in [19], but already Urysohn considered what can be called sequentially compact sequences [17]. In [8] it became clear that one needs to extend the concept of compactness to arbitrary families of sets in order to characterize open sets for the *Scott convergence* on the hyperspace of open sets (dually, the *upper Kuratowski convergence* on the hyperspace of closed sets). Later it turned out that compact families are essential in very useful characterizations of triquotient maps [16].

Let ξ be a convergence on a set X and let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of X. We say that $\mathcal{A}$ is ξ-*compact at* $\mathcal{B}$ if for every filter $\mathcal{H}$ on X,

$$\mathcal{H}\#\mathcal{A} \Longrightarrow \mathrm{adh}_\xi \mathcal{H} \in \mathcal{B}^\#. \tag{2.1}$$

[1] Some authors give additional conditions, like $\lim(\mathcal{F} \cap \mathcal{G}) \subset \lim \mathcal{F} \cap \lim \mathcal{G}$. I call such convergences *prototopologies*.
[2] If $\mathcal{B}$ is a filter-base on X, then we write $x \in \lim \mathcal{B}$ whenever $x \in \lim \mathcal{B}^\uparrow$, where $\mathcal{B}^\uparrow := \{F \subset X : \exists (B \in \mathcal{B})\ B \subset F\}$.
[3] That is, there is $P \in \mathcal{P}$ such that $P \in \mathcal{F}$.
[4] In particular, $\mathcal{P}$ is a cover of a topological space X if and only if $\bigcup_{P \in \mathcal{P}} \mathrm{int} P \supset X$.