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Linearized topologies and deformation theory ☆

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To my parents, with love, and with the joy of sharing the love of mathematics.

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ABSTRACT

In this paper, for an underlying small category $\mathcal{U}$ endowed with a Grothendieck topology τ, and a linear category $\mathfrak{a}$ which is graded over $\mathcal{U}$ in the sense of [13], we define a natural linear topology $\mathcal{T}_\tau$ on $\mathfrak{a}$, which we call the *linearized topology*. Grothendieck categories in (non-commutative) algebraic geometry can often be realized as linear sheaf categories over linearized topologies. With the eye on deformation theory, it is important to obtain such realizations in which the linear category contains a restricted amount of algebraic information. We prove several results on the relation between refinement (eliminating both objects, and, more surprisingly, morphisms) of the non-linear underlying site $(\mathcal{U}, \tau)$, and refinement of the linearized site $(\mathfrak{a}, \mathcal{T}_\tau)$. These results apply to several incarnations of (quasi-coherent) sheaf categories, leading to a description of the infinitesimal deformation theory of these categories in the sense of [17] which is entirely controlled by the Gerstenhaber deformation theory of the small linear category $\mathfrak{a}$, and the Grothendieck topology τ on $\mathcal{U}$. Our findings extend results from [17,12,7] and recover the examples from [21,20].

1. Introduction

In the 1960s, the Grothendieck school revolutionarized algebraic geometry by founding it on the theory of abelian categories, see [10], and on topos theory, see the SGA4 volumes, in particular [1]. The setup of scheme theory allows arbitrary commutative rings as building blocks, and is further centered around the concepts of (quasi-coherent) sheaves and sheaf cohomology. Schemes have underlying topological spaces, built from the Zariski topologies on the spectra of commutative rings. With the formulation of the Weil conjectures, it was realized that classical topological spaces and sheaf cohomology were insufficient, and it was the introduction of the more general étale Grothendieck topology, and corresponding étale cohomology, which eventually led to the proofs of the conjectures between 1960 and 1974. On the other hand, in 1962, in his thesis Gabriël

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developed localization theory in the context of abelian categories, involving, in the case of module categories over rings, the concept of a Gabriël filter on a ring. This notion can be recognized as a linear version of a Grothendieck topology, on a single object linear category, and can easily be extended to arbitrary small linear categories. In the famous Gabriël–Popescu theorem, it was proven that every Grothendieck abelian category can be realized as the localization of a module category. This gives Grothendieck categories the status of linear versions of Grothendieck topoi, the localizations of presheaf categories of sets which were characterized internally by Giraud's theorem. In both setups, the localizations can be realized as sheaf categories, and depend upon the choice of a suitable functor $\gamma : \mathfrak{a} \longrightarrow \mathcal{C}$ from a small (linear) category to a Grothendieck topos (or Grothendieck category), giving rise to a (linear) topology $\mathcal{T}$ on $\mathfrak{a}$ and an equivalence of categories $\mathcal{C} \cong \mathsf{Sh}(\mathfrak{a}, \mathcal{T})$. See [11] for a characterization of such functors γ.

Whereas the rings occurring in algebraic geometry are commutative, the theory of abelian categories and their localizations is not restricted to the commutative realm, and includes in particular module categories over non-commutative rings. For a commutative ring A, the module category $\mathsf{Mod}(A)$ is equivalent to the category of quasi-coherent sheaves on the spectrum $\mathrm{Spec}(A)$, and captures a lot of geometric information. With the development of so-called non-commutative algebraic geometry by Artin, Tate, Stafford, Van den Bergh and others [3,19], this observation is taken further and Grothendieck abelian categories are themselves considered as the main geometric objects. This is motivated by the fact that non-commutative rings typically have no well-behaved underlying "spectra" of points, whence one is forced to work in a point-free environment. Following this philosophy, one is primordially interested in Grothendieck categories which share a lot with the ones occurring in classical algebraic geometry. Examples are provided by deformations of commutative rings, with the Weyl algebra deforming the commutative polynomial algebra in two variables as prime example. Since, for instance, projective geometry involves more general quasi-coherent sheaf categories than module categories, in [17], Gerstenhaber's deformation theory of algebras was extended to a deformation theory for abelian categories. This theory allows to capture the important examples of non-commutative projective planes, quadrics and $\mathbb{P}_1$-bundles over commutative schemes from [21,20], which motivated its development. Further, the theory leads to a description of non-commutative deformations of schemes in terms of twisted presheaves of non-commutative rings (see [12]).

Let $\mathcal{C}$ be a given Grothendieck category over a field k, and suppose we are interested in deformation in the direction of an Artin local k-algebra R. According to [17], a deformation is an R-linear Grothendieck abelian category which reduces to $\mathcal{C}$ upon restriction to k-linear objects. Now suppose we consider our favorite representation $\mathcal{C} \cong \mathsf{Sh}(\mathfrak{a}, \mathcal{T})$ as a sheaf category over a k-linear site $(\mathfrak{a}, \mathcal{T})$, corresponding to a functor $\gamma : \mathfrak{a} \longrightarrow \mathcal{C}$. Then, ideally, we would like to realize $\mathcal{D}$ as $\mathcal{D} \cong \mathsf{Sh}(\mathfrak{b}, \mathcal{S})$ for an R-linear site $(\mathfrak{b}, \mathcal{S})$ in which:

(A) $\mathfrak{b}$ is obtained as a linear (i.e., Gerstenhaber type) deformation of $\mathfrak{a}$;
(B) $\mathcal{S}$ is naturally an "R-linear variant" of $\mathcal{T}$.

In general, both requirements may fail. Whether or not we can realize (A) essentially depends on homological conditions involving the objects $\gamma(A) \in \mathcal{C}$, more precisely the vanishing of certain Ext groups between these objects. In order to realize (B), we first have to understand what an R-linear variant of a k-linear topology means. In a first approach, this could mean "a topology naturally induced by $\mathcal{T}$ along the map $\mathfrak{b} \longrightarrow \mathfrak{a}$". The drawback of this interpretation is that such a topology does not necessarily have an intrinsic "meaning" with respect to $\mathfrak{b}$. Let us look at the ideal case where $\mathcal{C} = \mathsf{Mod}(\mathfrak{a})$, the entire module category over $\mathfrak{a}$. A basic result from [17] states that there is a deformation equivalence

$$\mathrm{Def}_{\mathrm{lin}}(\mathfrak{a}) \longrightarrow \mathrm{Def}_{\mathrm{ab}}(\mathsf{Mod}(\mathfrak{a})) : \mathfrak{b} \longrightarrow \mathsf{Mod}(\mathfrak{b}) \quad (1)$$