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Abstract sectional category in model structures on topological spaces

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ABSTRACT

We study the behavior of the abstract sectional category in the Quillen, the Strøm and the Mixed proper model structures on topological spaces and prove that, under certain reasonable conditions, all of them coincide with the classical notion. As a result, the same conclusions hold for the abstract Lusternik–Schnirelmann category and the abstract topological complexity of a space.

0. Introduction

It is well known that most numerical homotopy invariants of Lusternik–Schnirelmann type on topological spaces are derived from the *sectional category* or *genus* of a map, introduced by Schwarz in [15]. For a given map $f\colon X \to Y$, its sectional category $\mathbf{secat}(f)$ is the smallest integer n such that Y can be covered by $n+1$ open subsets on each of which f admits a homotopy local section. As this invariant appears in different settings, and in order to study it under a common viewpoint, in [4] the authors introduced the abstract sectional category in a J-category, for instance a (proper) closed model category satisfying the so called cube lemma (see next sections for precise definitions). This is done via two different notions, the *Ganea sectional category* $\mathrm{Gsecat}(f)$ and the *Whitehead sectional category* $\mathrm{Wsecat}(f)$ of a morphism f on a general model category $\mathcal{C}$, which are shown to coincide, in a J-category [4, Thm. 15]. Both equivalent invariants are denoted simply by $\mathrm{secat}(f)$, or $\mathrm{secat}^{\mathcal{C}}(f)$ if we want to stress in which category we are working. Then, in the

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topological setting, the classical sectional category of a map is precisely its abstract sectional category on the model structure of topological spaces in the sense of Strøm [17] (see next sections for precise definitions).

Our main goal is to show that, under sufficiently general conditions, one can also recover this invariant from the abstract sectional category on the original Quillen model category structure [14]. Indeed, from now on, denote indistinctly by $\mathcal{T}$ either the category **Top** or $\mathbf{Top}_*$ of free and pointed topological spaces. Also, denote by $\mathcal{T}_S$, and $\mathcal{T}_Q$ the Strøm and Quillen model structure respectively in $\mathcal{T}$. Then, we prove (see Theorem 2.5 for a more precise statement):

Theorem 0.1. *Let $f\colon X \to Y$ be a map in which X, Y are of the homotopy type of locally compact CW-complexes. Then,*

$$\operatorname{secat}^{\mathcal{T}_S}(f) = \operatorname{secat}^{\mathcal{T}_Q}(f).$$

In particular, in the free setting and with Y normal, any of these invariant yields $\mathbf{secat}(f)$.

As a consequence, we show that these common invariants have a simplicial, and therefore, combinatorial description. Indeed, let $\mathcal{SS}\mathrm{et}$ denote the category of either free or pointed simplicial sets. Using the classical Milnor equivalence [12] we deduce the following (see the first assertion of Theorem 2.11 for a precise and more general statement).

Theorem 0.2. *Let $f\colon X \to Y$ be a map in which Y is normal and X, Y have the homotopy type of locally compact CW-complexes. Then,*

$$\mathbf{secat}(f) = \operatorname{secat}^{\mathcal{SS}\mathrm{et}}\bigl(\operatorname{Sing}(f)\bigr).$$

These results readily translate to widely studied LS invariants. Recall that the *Lusternik–Schnirelmann category* $\mathbf{cat}(X)$ of a given topological space [10] is the least integer n for which X can be covered by $n+1$ open sets deformable to a point within X. On the other hand, the *topological complexity* $\mathbf{TC}(X)$ of X [6] is the least integer n such that $X \times X$ can be covered by $n+1$ open sets on each of which there is a section of the path fibration $X^I \to X \times X$ which associates to each path its initial and final points. When X is the configuration space associated to the motion of a given mechanical system, this invariant measures, roughly speaking, the minimum amount of instructions needed for controlling the given system.

In a J-category $\mathcal{C}$, the *abstract Lusternik–Schnirelmann category* $\operatorname{cat}^{\mathcal{C}}(X)$ of a given object X is defined as $\operatorname{secat}^{\mathcal{C}}(* \to X)$ where $*$ denotes the 0-object of $\mathcal{C}$ [5]. In the same way, the *abstract topological complexity* $\operatorname{TC}^{\mathcal{C}}(X)$ is defined as $\operatorname{secat}^{\mathcal{C}}(\Delta)$ where $\Delta\colon X \to X \times X$ is the diagonal [4]. Then, the path fibration is homotopy equivalent to the diagonal, and for path-connected spaces one has

$$\mathbf{cat}(X) = \operatorname{cat}^{\mathcal{T}_S}(X), \qquad \mathbf{TC}(X) = \operatorname{TC}^{\mathcal{T}_S}(X).$$

For the first equality X only needs to be normal and well pointed [3, Thm. 1.55], and $\mathcal{T} = \mathbf{Top}_*$. For the second, $\mathcal{T} = \mathbf{Top}$ and we require $X \times X$ to be normal [7, Thm. 2.2]. Under these general hypotheses, the theorems above immediately imply (this is Corollary 2.8 and the second assertion of Theorem 2.11):

Corollary 0.3. *For any space X of the homotopy type of a locally compact CW-complex,*

$$\mathbf{TC}(X) = \operatorname{TC}^{\mathcal{T}_Q}(X) = \operatorname{TC}^{\mathcal{SS}\mathrm{et}}\bigl(\operatorname{Sing}(X)\bigr),$$
$$\mathbf{cat}(X) = \operatorname{cat}^{\mathcal{T}_Q}(X) = \operatorname{cat}^{\mathcal{SS}\mathrm{et}}\bigl(\operatorname{Sing}(X)\bigr).$$