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Ramification of the Gauss map and the total curvature of a complete minimal surface

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ABSTRACT

In this article, we study the relations between the ramifications of the Gauss map and the total curvature of a complete minimal surface. More precisely, we introduce some conditions on the ramifications of the Gauss map of a complete minimal surface M to show that M has finite total curvature.

1. Introduction

In 1988, Fujimoto [3] proved Nirenberg's conjecture that if M is a complete non-flat minimal surface in $\mathbb{R}^3$, then its Gauss map can omit at most 4 points, and there are a number of examples showing that the bound is sharp (see [12, pp. 72–74]). He [4] also extended that result to the Gauss map of complete minimal surfaces in $\mathbb{R}^m$. After that, in 1990, Mo–Osserman [10] showed an interesting improvement of Fujimoto's result by proving that a complete minimal surface in $\mathbb{R}^3$ whose Gauss map assumes five values only a finite number of times has finite total curvature. We note that a complete minimal surface with finite total

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curvature to be called an algebraic minimal surface. After that, Mo [9] extended that result to the complete minimal surface in $\mathbb{R}^m$ $(m > 3)$.

On the other hand, in 1993, M. Ru [13] refined the results of Fujimoto by studying the Gauss map of minimal surfaces in $\mathbb{R}^m$ with ramification. Many results related to this problem were studied (see Jin–Ru [7], Kawakami–Kobayashi–Miyaoka [8], Ha [5], Dethloff–Ha [1] and Dethloff–Ha–Thoan [2] for examples).

A natural question is whether we may show a relation between of the ramification of the Gauss map and the total curvature of a complete minimal surface. The main purpose of this article is to give an affirmative answer for this question. For the purpose of this article, we recall some definitions.

Let $x = (x_0, \cdots, x_{m-1}) : M \to \mathbb{R}^m$ be a (smooth, oriented) minimal surface immersed in $\mathbb{R}^m$. Then M has the structure of a Riemann surface and any local isothermal coordinate (ξ_1, ξ_2) of M gives a local holomorphic coordinate $z = \xi_1 + \sqrt{-1}\xi_2$. The (generalized) Gauss map of x is defined to be

$$g : M \to Q_{m-2}(\mathbb{C}) \subset \mathbb{P}^{m-1}(\mathbb{C}), g(z) = (\frac{\partial x_0}{\partial z} : \cdots : \frac{\partial x_{m-1}}{\partial z}),$$

where

$$Q_{m-2}(\mathbb{C}) = \{(w_0 : \cdots : w_{m-1}) | w_0^2 + \cdots + w_{m-1}^2 = 0\} \subset \mathbb{P}^{m-1}(\mathbb{C}).$$

By the assumption of minimality of M, g is a holomorphic map of M into $Q_{m-2}(\mathbb{C})$.

One says that g is ramified over a hyperplane $H = \{(w_0 : \cdots : w_{m-1}) \in \mathbb{P}^{m-1}(\mathbb{C}) : a_0w_0 + \cdots + a_{m-1}w_{m-1} = 0\}$ *with multiplicity at least* e if all the zeros of the function $(g, H) := a_0g_0 + \cdots + a_{m-1}g_{m-1}$ have orders at least e, where $g = (g_0 : \cdots : g_{m-1})$. If the image of g omits H, one will say that g is *ramified over* H *with multiplicity* ∞.

The main purpose of this article is to prove the following:

Theorem 1. *Let M be a complete minimal surface in $\mathbb{R}^m$ and K be a compact subset in M. Assume that the generalized Gauss map g of M is k-non-degenerate (that is $g(M)$ is contained in a k-dimensional linear subspace in $\mathbb{P}^{m-1}(\mathbb{C})$, but none of lower dimension), $1 \le k \le m-1$. If there are q hyperplanes $\{H_j\}_{j=1}^q$ in N-subgeneral position in $\mathbb{P}^{m-1}(\mathbb{C})$, ($N \ge m-1$) such that g is ramified over H_j with multiplicity at least m_j on $M \setminus K$ for each j and*

$$\sum_{j=1}^{q}(1 - \frac{k}{m_j}) > (k+1)(N - \frac{k}{2}) + (N+1), \tag{1.1}$$

then M has finite total curvature.

In particular, if $\{H_j\}_{j=1}^q$ are in general position in $\mathbb{P}^{m-1}(\mathbb{C})$ and

$$\sum_{j=1}^{q}(1 - \frac{m-1}{m_j}) > \frac{m(m+1)}{2}, \tag{1.2}$$

then M must have finite total curvature.

When $m = 3$, we can identify $\mathbb{Q}_1(\mathbb{C})$ with $\mathbb{P}^1(\mathbb{C})$. So we can get a better result as the following:

Theorem 2. *Let M be a complete minimal surface in $\mathbb{R}^3$ and q distinct points $a^j, \ldots, a^q$ in $\mathbb{P}^1(\mathbb{C})$. Suppose that the Gauss map g of M is ramified over a^j with multiplicity at least m_j for each $j = 1, \cdots, q$ outside a compact subset K of M. Then M has finite total curvature if*