Topology and its Applications 199 (2016) 55–62

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On Baire class one functions on a product space

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ARTICLE INFO

Article history:
Received 24 May 2015
Received in revised form 25 November 2015
Accepted 8 December 2015
Available online 17 December 2015

MSC:
54C30
26A21

Keywords:
Baire class
Tychonoff topology
Semicontinuous function

ABSTRACT

We show that every Baire class one function on a countable product of metric spaces is the pointwise limit of a sequence of continuous functions, each depending only on finitely many coordinates of the argument. It is proved also that this result does not extend to higher Baire classes (with continuous functions replaced by arbitrary ones).

1. Introduction

Let $\mathcal{X}$ be the Tychonoff product of a countable collection of non-singleton completely regular spaces $\mathcal{X}_j$, $j \in \mathbb{N}$, S be a subset of $\mathcal{X}$, and Y be a metrizable space.

We call a function $f: S \to Y$ *finitely determined* if it depends only on finitely many coordinates of its argument. More precisely, f is finitely determined if there exists an $m \in \mathbb{N}$ such that $f(x) = f(y)$ for each pair of points $x, y \in S$ whose first m coordinates coincide. Let us denote by $CF(S, Y)$ the set of all continuous finitely determined functions from S to Y. For brevity, let us write $CF(S)$ for $CF(S, \mathbb{R})$.

Let us remark that every function $f: S \to \mathbb{R}$ is the pointwise limit of a net in $CF(S)$. Indeed, for any finite set $A \subset S$, there is a function in $CF(S)$ which agrees with f on A, so $CF(S)$ is dense in the set of all functions f with the pointwise topology.

The questions naturally arise whether 1) every continuous function from $\mathcal{X}$ to Y is the uniform limit of a sequence in $CF(\mathcal{X}, Y)$ and whether 2) every real-valued function of Baire class $\alpha \geq 1$ on S is the pointwise limit of a sequence of finitely determined functions of previous Baire classes.

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http://dx.doi.org/10.1016/j.topol.2015.12.003

Recall the definition of Baire classes. For a given topological space X, let $B_0(X)$ be the set of all continuous real-valued functions on X, and inductively define $B_\alpha(X)$ for each countable ordinal α to be the set of pointwise limits of sequences in $\bigcup_{\xi<\alpha} B_\xi(X)$.

We shall prove that 1) holds if and only if $\mathcal{X}$ is pseudocompact and that for every metrizable $\mathcal{X}$ 2) holds for any $S \subset \mathcal{X}$ if and only if $\alpha = 1$. It is worth noting that the "if" part of the former assertion for the case of real-valued functions on a compact $\mathcal{X}$ is well known [7]; it is a consequence of the Stone–Weierstrass Theorem [2, XIII.3.3], since $CF(\mathcal{X})$ is a point-separating unitary algebra.

At the end of the paper we show that for a lower semicontinuous function on a metrizable $\mathcal{X}$ to be the pointwise limit of an increasing sequence in $CF(\mathcal{X})$ it is necessary and sufficient that all but finitely many of the factors $\mathcal{X}_j$ be compact.

For a treatment of functions on an uncountable Tychonoff product we refer the reader to [3].

2. Uniform approximation

Theorem 1. *The following statements are equivalent:*

a) $\mathcal{X}$ *is pseudocompact;*
b) *for every metrizable space* Y*, every continuous function* $f: \mathcal{X} \to Y$ *is the uniform limit of a sequence in* $CF(\mathcal{X}, Y)$*;*
c) *every continuous function* $f: \mathcal{X} \to \mathbb{R}$ *is the uniform limit of a sequence in* $CF(\mathcal{X})$*.*

To prove Theorem 1 we shall need two lemmas, the first of which is well known.

Throughout the following, d is a compatible metric on Y. Let X be a topological space. Recall that a sequence $\{f_n\}$ of functions from X to Y is said to converge to a function $f: X \to Y$ uniformly at x provided that for every $\varepsilon > 0$ there are a neighbourhood U_x of x and number N such that $d(f_n(y), f(y)) < \varepsilon$ whenever $y \in U_x$ and $n \geq N$.

Lemma 1. *Let* X *be a pseudocompact space. If a sequence of continuous functions from* X *to* Y *converges uniformly at each point of* X*, then it converges uniformly on* X*.*

Proof. Let $\{f_n\}$ be a sequence of continuous functions from X to Y that converges to f uniformly at each point of X. Then f is also continuous [4, §42.IV]. Let $g_n(x) = d(f_n(x), f(x))$ for all $x \in X$ and $n \in \mathbb{N}$. It follows that the sequence $\{g_n\}$ of real-valued continuous functions converges to zero uniformly at each point of X. By [1, Theorem 2], the sequence $\{g_n\}$ converges to zero uniformly on X, which is equivalent to what is to be proved. □

For the rest of the paper, fix $x^0 \in \mathcal{X}$ and for each $n \in \mathbb{N}$ define p_n to be the function which maps a point $x \in \mathcal{X}$ into the point with the first n coordinates equal to those of x and the other coordinates equal to those of x^0.

Lemma 2. *Let* $f: \mathcal{X} \to Y$ *be a continuous function. Then the sequence* $\{f \circ p_n\}$ *converges to* f *uniformly at each point of* $\mathcal{X}$*. Moreover,* f *is the uniform limit of a sequence in* $CF(\mathcal{X}, Y)$ *if and only if the sequence* $\{f \circ p_n\}$ *converges to* f *uniformly on* $\mathcal{X}$*.*

Proof. Let $x \in \mathcal{X}$ and $\varepsilon > 0$ be given. By virtue of continuity of f there is a neighbourhood U of x such that $d(f(x), f(y)) < \varepsilon/2$ for all $y \in U$. By definition of the product topology, there exists a basic open set