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A partial classification of inverse limits with irreducible functions

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ABSTRACT

We discuss a class of upper semi-continuous set-valued functions called irreducible functions, and we develop multiple tools for distinguishing topologically between their inverse limits. First, we discuss properties of subcontinua of their inverse limits, and we use those properties to show that for certain irreducible functions, given two, if their graphs contain different (finite) numbers of maximal nowhere dense arcs, then they have topologically distinct inverse limits. Additionally, we discuss endpoints of inverse limits with irreducible functions, and finally, we apply these tools to obtain a complete classification of the inverse limits arising from four specific families of irreducible functions.

0. Introduction

Inverse limits with upper semi-continuous set-valued functions were first introduced by Mahavier in [7], and they were further developed by Ingram and Mahavier in [2]. One particular area of study with such inverse limits has been indecomposability (see [1,5,6,10,11]). All of the set-valued functions considered in this paper have indecomposable continua as their inverse limits. More specifically, all of the functions considered are irreducible functions, as defined in [5]. An irreducible function is defined by its inverse which is the union of continuous single-valued maps (with certain properties).

In [5], a sufficient condition was presented for the inverse limits of two irreducible functions to be homeomorphic. This condition is not necessary, however, which might lead one to ask when two irreducible functions would have topologically distinct inverse limits. A partial answer to this question was presented in [3] where chainability was characterized for inverse limits with irreducible functions on the unit interval.

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In this paper, we develop additional tools which may be applied to determine when two inverse limits are not homeomorphic, and we use these tool to establish a partial classification of inverse limits with irreducible functions on $[0, 1]$. Specifically, in Section 2, we look at a specific subclass of irreducible functions, and we show that for a function F in this subclass, every proper subcontinuum of the inverse limit of F is homeomorphic to a subcontinuum of the set

$$\{\mathbf{x} \in X^n : x_i \in F_i(x_{i+1}) \text{ for } 1 \leq i < n\}$$

for some $n \in \mathbb{N}$. This result is stated in Theorem 2.3.

Then in Section 3, we apply this result to irreducible functions on $[0, 1]$. We show that if the graphs of two irreducible functions have different (finite) numbers of maximal nowhere dense arcs, and the one with more satisfies the hypotheses of Theorem 2.3, then the two inverse limits are not homeomorphic.

Next, in Section 4, we consider endpoints of inverse limits. A characterization was given in [4] for endpoints of inverse limits with set-valued functions whose inverse was the union of mappings. In particular, any irreducible function has its inverse equal to a union of mappings, and we show how this characterization can be used to determine precisely which points of the inverse limit are endpoints for certain irreducible functions.

Finally, in Section 5, we define four specific families of irreducible functions on the unit interval. We implement all of the results from the previous sections in order to give a complete classification of the inverse limits of the functions from these four families.

1. Background definitions and theorems

A set X is a *continuum* if it is a non-empty, compact, connected subset of a metric space. A subset of a continuum X which is itself a continuum is called a *subcontinuum* of X. Given a continuum X and a point $p \in X$, we say that p is an *endpoint of* X if any two subcontinua of X which both contain p are nested. Given a continuum X and two points $a, b \in X$, we say that X is *irreducible between* a *and* b if no proper subcontinuum of X intersects both a and b. More generally, if $A, B \subseteq X$ are closed in X, we say that X is *irreducible between* A *and* B if no proper subcontinuum of X intersects both A and B. A continuum is called *irreducible* if it is irreducible between some two points.

If X is a continuum, we denote by 2^X the set of all non-empty compact subsets of X. The *graph* of a function $F : X \to 2^Y$ is the set

$$\Gamma(F) = \{(x, y) \in X \times Y : y \in F(x)\}.$$

A function $F : X \to 2^Y$ is called *upper semi-continuous* if its graph is a closed subset of $X \times Y$. (This is not the standard definition, but it was shown in [2] that this is equivalent to the standard definition when X and Y are compact Hausdorff spaces.)

Suppose $\mathbf{X} = (X_i)_{i\in\mathbb{N}}$ is a sequence of continua, and $\mathbf{F} = (F_i)_{i\in\mathbb{N}}$ is a sequence of upper semi-continuous functions such that for each $i \in \mathbb{N}$, $F_i : X_{i+1} \to 2^{X_i}$. Then the pair $\{\mathbf{X}, \mathbf{F}\}$ is called an *inverse sequence*, and the *inverse limit* of that inverse sequence, denoted $\varprojlim \mathbf{F}$, is the set

$$\varprojlim \mathbf{F} = \{\mathbf{x} \in \prod_{i=1}^{\infty} X_i : x_i \in F_i(x_{i+1}) \text{ for all } i \in \mathbb{N}\}.$$

(In this paper, a sequence—either finite or infinite—will be written as a bold letter, and its terms will be written using the same letter, subscripted and italicized.) The continua, X_i, are called the *factor spaces* of the inverse sequence; and the upper semi-continuous functions, F_i, are called the *bonding functions* of the