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A sharp representation of multiplicative isomorphisms of uniformly continuous functions

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ABSTRACT

Let X, Y be complete metric spaces, $U(X)$, $U(Y)$ the spaces of uniformly continuous functions with real values defined on X and Y. We will show the form that every multiplicative isomorphism $T : U(Y) \to U(X)$ has: for x outside a uniformly isolated subset $S \subset X$,

$$Tf(x) = \mathrm{sign}(f(\tau(x)))|f(\tau(x))|^{1+p(x)},$$

where $\tau : X \to Y$ is a uniform homeomorphism and $p : X \setminus S \to \mathbb{R}$ is such that $p \cdot h \cdot \log h$ is uniformly continuous for every $h \in U(X, (2, \infty))$.

0. Introduction

In this paper, we will consider the space $U(X)$ endowed with pointwise multiplication. Our main interest lies in bijections preserving multiplication in both directions: $T : U(Y) \to U(X)$ such that $T(f \cdot g) = Tf \cdot Tg$ provided $f, g, f \cdot g \in U(Y)$ – analogously for T^{-1}.

We will show that these bijections behave the best way they could: as every map $f : X \to \mathbb{R}$ is uniformly continuous when X is uniformly isolated, we cannot *control* what happens in $S \subset X$ when $d(s, x) \geq \varepsilon$ for every $s \in S$, $x \in X$, $x \neq s$. Outside a uniformly isolated S, we can ensure $Tf = \mathrm{sign}(f \circ \tau)|f \circ \tau|^{1+p}$, where $\tau : X \to Y$ is a uniform homeomorphism and $p : X \setminus S \to \mathbb{R}$ is a very small function: $p \cdot h \cdot \log h \in U(X)$ whenever $h \in U(X)$, $h \geq 2$. Of course, this implies that if T is a linear and multiplicative bijection, then $Tf = f \circ \tau$ for every $f \in U(Y)$.

Antecessors.

As far as we know, the first result in which the multiplicative structure of a space of functions is used to determine the topological structure of the underlying space is due to Gel'fand and Kolmogorov. They proved

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in [7] that, provided X and Y are compact Hausdorff topological spaces, $C(X)$ and $C(Y)$ are isomorphic algebras if and only if X and Y are homeomorphic.

The next step in characterizing the topological structure of a compact space was Milgram's paper [8], where he *relaxes* the hypothesis in the previous theorem, showing X and Y must be homeomorphic whenever $C(X)$ and $C(Y)$ are isomorphic as multiplicative semigroups. He also gives a representation of these semigroup isomorphisms. Namely, he proves that for every isomorphism $T : C(Y) \to C(X)$, there exists a (possibly empty) finite subset $S \subset X$ and a continuous function $p : X \setminus S \to \mathbb{R}$ such that

$$Tf(x) = \operatorname{sign}(f(\tau(x)))|f(\tau(x))|^{p(x)}$$

for every $f \in C(Y)$, $x \in X \setminus S$.

Császár, in [5], generalized this by showing that every multiplicative isomorphism $T : C(Y) \to C(X)$ induces a homeomorphism $\tau : X \to Y$, provided X and Y are realcompact Tychonoff spaces.

Dealing with uniformly continuous functions, in [1] it is shown that, given complete metric spaces X, Y, and $\mathbb{I} = [0, 1]$, every multiplicative isomorphism $T : U(Y, \mathbb{I}) \to U(X, \mathbb{I})$ has the form

$$Tf = (f \circ \tau)^t,$$

where $\tau : X \to Y$ is a uniform homeomorphism and $t : X \to (0, \infty)$ is uniformly continuous and, outside a uniformly isolated subset $S \subset X$, it is bounded. The main result in this paper is also related to that in [3], where we showed that every lattice isomorphism $T : U(Y) \to U(X)$ induces a uniform homeomorphism $\tau : X \to Y$.

A nice review about this kind of results can be found in [6].

Plan of the paper.

We will consider two complete metric spaces X and Y and a multiplicative isomorphism $T : U(Y) \to U(X)$.

In the first section, we will state the basic definitions and properties. We will also use some results stated in [2] and [3] to prove that there exists a uniform homeomorphism $\tau : X \to Y$ such that $Tf(x)$ depends only on x and on the value that f takes in $\tau(x)$.

In the second section, we show the existence of a uniformly isolated $S \subset X$ and a uniformly continuous $p : X \setminus S \to (0, \infty)$ such that $Tf(x) = \pm|f(\tau(x))|^{p(x)}$ for every $f \in U(Y)$, $x \in X \setminus S$ (compare with the representation given by Milgram). Later, we characterize this multiplicative isomorphisms in terms of p.

In the third and last section we will put some illustrative examples and consequences of the main result.

1. First steps

We will consider complete metric spaces X and Y and their spaces of uniformly continuous functions with real values, $U(X)$, $U(Y)$, endowed with pointwise product.

1.1. A map $T : U(Y) \to U(X)$ is a multiplicative isomorphism if it is a bijection such that

- $T(f \cdot g) = (Tf) \cdot (Tg)$ whenever $f, g, f \cdot g \in U(Y)$.
- $T^{-1}(f \cdot g) = (T^{-1}f) \cdot (T^{-1}g)$ whenever $f, g, f \cdot g \in U(X)$.

From now on, we will consider an isomorphism $T : U(Y) \to U(X)$ fixed.

1.2. Let $0 \neq c \in \mathbb{R}, p \in (0, \infty)$. For the sake of simplicity, we will write $[c]^p$ instead of $\operatorname{sign}(c)|c|^p$; $[0]^p = 0$. Given $f : X \to \mathbb{R}, p : X \to (0, \infty)$, $[f]^p$ is the function $x \mapsto [f(x)]^{p(x)}$.