Topology and its Applications 197 (2016) 21–27

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Semi-stratifiable spaces with monotonically normal compactifications

Harold Bennett [a], David Lutzer [b,*]

[a] Texas Tech University, Lubbock, TX 79409, United States
[b] College of William and Mary, Williamsburg, VA 23187, United States

ARTICLE INFO

Article history:
Received 6 August 2015
Accepted 13 October 2015
Available online 11 November 2015

MSC:
primary 54D15
secondary 54D30, 54D35, 54F05

Keywords:
Monotonically normal space
Monotonically normal compactification
Rudin's solution of Nikiel's problem
Metrizable space
Countable subspace
σ-Discrete space
Semi-stratifiable
Stratifiable
Scattered
Dense metrizable subset

ABSTRACT

In this paper we use Mary Ellen Rudin's solution of Nikiel's problem to investigate metrizability of certain subsets of compact monotonically normal spaces. We prove that if H is a semi-stratifiable space that can be covered by a σ-locally-finite collection of closed metrizable subspaces and if H embeds in a monotonically normal compact space, then H is metrizable. It follows that if H is a semi-stratifiable space with a monotonically normal compactification, then H is metrizable if it satisfies any one of the following: H has a σ-locally finite cover by compact subsets; H is a σ-discrete space; H is a scattered; H is σ-compact. In addition, a countable space X has a monotonically normal compactification if and only if X is metrizable. We also prove that any semi-stratifiable space with a monotonically normal compactification is first-countable and is the union of a family of dense metrizable subspaces. Having a monotonically normal compactification is a crucial hypothesis in these results because R.W. Heath has given an example of a countable non-metrizable stratifiable (and hence monotonically normal) group. We ask whether a first-countable semi-stratifiable space must be metrizable if it has a monotonically normal compactification. This is equivalent to "If X is a first-countable stratifiable space with a monotonically normal compactification, must H be metrizable?"

1. Introduction

Research since the 1970s shows that there are close parallels between generalized ordered (GO-) spaces and monotonically normal spaces, particularly in the theories of cardinal functions and of paracompactness (see [1]). In this paper we investigate the extent to which metrization theory for subsets of compact monotonically normal spaces resembles metrization theory for GO-spaces, i.e., for subspaces of compact linearly ordered spaces.

One of the most celebrated results in recent set-theoretic topology is Mary Ellen Rudin's solution of Nikiel's problem [17] and it is our primary tool in this study. Rudin proved:

* Corresponding author.
E-mail address: lutzer@math.wm.edu (D. Lutzer).

http://dx.doi.org/10.1016/j.topol.2015.10.005

Theorem 1.1. *Any compact monotonically normal space is the continuous image of a compact linearly ordered topological space.*

In Section 4 of this paper we combine ordered space techniques with Theorem 1.1 to prove:

Proposition 1.2. *Suppose H is a subspace of a compact monotonically normal space (equivalently, suppose H has a monotonically normal compactification). Then:*

a) *there is a GO-space $(Z, \tau, \preceq)$ and a perfect irreducible mapping g from Z onto H with the property that no fiber of g contains a jump of $(Z, \preceq)$ (see (4.2));*
b) *if each point of H is a G_δ-set, then each fiber of the mapping g is metrizable (see (4.3));*
c) *if H has a G_δ-diagonal, then so does Z (see (4.4)).*

We use Proposition 1.2 to study which metrization theorems for GO-spaces can be extended to the larger category of spaces with monotonically normal compactifications.

The basic metrization theorem for GO-spaces that might generalize to monotonically normal spaces (because it does not mention the order of the GO-space) appears in [15]:

Theorem 1.3. *A GO-space X is metrizable if and only if X is semi-stratifiable.*

Easy examples show that Theorem 1.3 does not generalize to arbitrary monotonically normal spaces because there are many non-metrizable spaces that are stratifiable [2,3,10], and stratifiable spaces are exactly the semi-stratifiable monotonically normal spaces. However, because every GO-space has a GO-compactification (which, of course, is monotonically normal), it is natural to ask the following more interesting question:

Question 1. Suppose a space H is semi-stratifiable and has a monotonically normal compactification. Must H be metrizable?

As a preliminary step toward that question, in this paper we prove the following result.

Theorem 1.4. *Suppose H is a semi-stratifiable space with a monotonically normal compactification. Then:*

a) *the space H is the union of a family of dense metrizable subspaces (see (3.4));*
b) *the space H is first-countable (see (3.4)).*

In addition, H is metrizable if any one of the following holds:

c) *if there is a σ-locally finite cover of H by closed metrizable subsets (see (3.1));*
d) *if there is a σ-locally-finite cover of H by compact subsets (see 3.2(a));*
e) *if $H = \bigcup\{H_n : n \geq 1\}$ where each H_n is a closed discrete subset[1] of H (see 3.2(b));*
f) *if H is scattered (see 3.2(c));*
g) *if H is σ-compact (see 3.2(d));*
h) *if H is countable (see 3.2(e)).*

[1] In the literature, spaces that are countable unions of closed discrete subspaces are called "σ-discrete spaces." If a σ-discrete space H has a monotonically normal compactification, then it must be stratifiable, so that a theorem of Gruenhage [8] shows that H must be at least M_1. Our result shows that if a σ-discrete space H has a monotonically normal compactification, then H is even more than M_1.