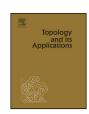


Contents lists available at ScienceDirect

## Topology and its Applications

www.elsevier.com/locate/topol



# Semi-stratifiable spaces with monotonically normal compactifications



Harold Bennett<sup>a</sup>, David Lutzer<sup>b,\*</sup>

- <sup>a</sup> Texas Tech University, Lubbock, TX 79409, United States
- <sup>b</sup> College of William and Mary, Williamsburg, VA 23187, United States

#### ARTICLE INFO

#### Article history: Received 6 August 2015 Accepted 13 October 2015 Available online 11 November 2015

MSC: primary 54D15 secondary 54D30, 54D35, 54F05

Keywords: Monotonically normal space Monotonically normal compactification Rudin's solution of Nikiel's problem Metrizable space Countable subspace  $\sigma$ -Discrete space Semi-stratifiable Stratifiable Scattered Dense metrizable subset

#### ABSTRACT

In this paper we use Mary Ellen Rudin's solution of Nikiel's problem to investigate metrizability of certain subsets of compact monotonically normal spaces. We prove that if H is a semi-stratifiable space that can be covered by a  $\sigma$ -locally-finite collection of closed metrizable subspaces and if H embeds in a monotonically normal compact space, then H is metrizable. It follows that if H is a semi-stratifiable space with a monotonically normal compactification, then H is metrizable if it satisfies any one of the following: H has a  $\sigma$ -locally finite cover by compact subsets; H is a  $\sigma$ -discrete space; H is a scattered; H is  $\sigma$ -compact. In addition, a countable space X has a monotonically normal compactification if and only if X is metrizable. We also prove that any semi-stratifiable space with a monotonically normal compactification is first-countable and is the union of a family of dense metrizable subspaces. Having a monotonically normal compactification is a crucial hypothesis in these results because R.W. Heath has given an example of a countable non-metrizable stratifiable (and hence monotonically normal) group. We ask whether a firstcountable semi-stratifiable space must be metrizable if it has a monotonically normal compactification. This is equivalent to "If X is a first-countable stratifiable space with a monotonically normal compactification, must H be metrizable?"

© 2015 Elsevier B.V. All rights reserved.

#### 1. Introduction

Research since the 1970s shows that there are close parallels between generalized ordered (GO-) spaces and monotonically normal spaces, particularly in the theories of cardinal functions and of paracompactness (see [1]). In this paper we investigate the extent to which metrization theory for subsets of compact monotonically normal spaces resembles metrization theory for GO-spaces, i.e., for subspaces of compact linearly ordered spaces.

One of the most celebrated results in recent set-theoretic topology is Mary Ellen Rudin's solution of Nikiel's problem [17] and it is our primary tool in this study. Rudin proved:

E-mail address: lutzer@math.wm.edu (D. Lutzer).

<sup>\*</sup> Corresponding author.

**Theorem 1.1.** Any compact monotonically normal space is the continuous image of a compact linearly ordered topological space.

In Section 4 of this paper we combine ordered space techniques with Theorem 1.1 to prove:

**Proposition 1.2.** Suppose H is a subspace of a compact monotonically normal space (equivalently, suppose H has a monotonically normal compactification). Then:

- a) there is a GO-space  $(Z, \tau, \preceq)$  and a perfect irreducible mapping g from Z onto H with the property that no fiber of g contains a jump of  $(Z, \preceq)$  (see (4.2));
- b) if each point of H is a  $G_{\delta}$ -set, then each fiber of the mapping g is metrizable (see (4.3));
- c) if H has a  $G_{\delta}$ -diagonal, then so does Z (see (4.4)).

We use Proposition 1.2 to study which metrization theorems for GO-spaces can be extended to the larger category of spaces with monotonically normal compactifications.

The basic metrization theorem for GO-spaces that might generalize to monotonically normal spaces (because it does not mention the order of the GO-space) appears in [15]:

**Theorem 1.3.** A GO-space X is metrizable if and only if X is semi-stratifiable.

Easy examples show that Theorem 1.3 does not generalize to arbitrary monotonically normal spaces because there are many non-metrizable spaces that are stratifiable [2,3,10], and stratifiable spaces are exactly the semi-stratifiable monotonically normal spaces. However, because every GO-space has a GO-compactification (which, of course, is monotonically normal), it is natural to ask the following more interesting question:

**Question 1.** Suppose a space H is semi-stratifiable and has a monotonically normal compactification. Must H be metrizable?

As a preliminary step toward that question, in this paper we prove the following result.

**Theorem 1.4.** Suppose H is a semi-stratifiable space with a monotonically normal compactification. Then:

- a) the space H is the union of a family of dense metrizable subspaces (see (3.4));
- b) the space H is first-countable (see (3.4)).

In addition, H is metrizable if any one of the following holds:

- c) if there is a  $\sigma$ -locally finite cover of H by closed metrizable subsets (see (3.1));
- d) if there is a  $\sigma$ -locally-finite cover of H by compact subsets (see 3.2(a));
- e) if  $H = \bigcup \{H_n : n \geq 1\}$  where each  $H_n$  is a closed discrete subset of H (see 3.2(b));
- f) if H is scattered (see 3.2(c));
- g) if H is  $\sigma$ -compact (see 3.2(d));
- h) if H is countable (see 3.2(e)).

<sup>&</sup>lt;sup>1</sup> In the literature, spaces that are countable unions of closed discrete subspaces are called "σ-discrete spaces." If a σ-discrete space H has a monotonically normal compactification, then it must be stratifiable, so that a theorem of Gruenhage [8] shows that H must be at least  $M_1$ . Our result shows that if a σ-discrete space H has a monotonically normal compactification, then H is even more than  $M_1$ .

### Download English Version:

# https://daneshyari.com/en/article/4658109

Download Persian Version:

https://daneshyari.com/article/4658109

<u>Daneshyari.com</u>